Mastermath Exam Set Theory
25-06-2013; 16:00–19:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–16.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

(2) 1. Given our definitions of ordered pairs ($\langle x, y\rangle = \{\{x\}, \{x, y\}\}$) and natural numbers ($n = \{0, \ldots, n-1\}$), which of the following is true:

A. $\langle 1, 2\rangle \subseteq 3$

⇒ **B.** $\{1, 2\} \subseteq 3$ $\{1, 2 \in 3\}$

C. $3 \subseteq \langle 1, 2\rangle$

D. $3 \subseteq \{1, 2\}$

(2) 2. Consider the structure $\langle \mathbb{N}, <\rangle$, for the language of set theory. Which of the following axioms of ZF *does not hold* in this structure.

A. Power Set

⇒ **B.** Pairing $\{$there is no x in $\mathbb{N}$ such that $(\forall y)(y < x \leftrightarrow (y = 3 \vee y = 4))\}$

C. Regularity

D. Extensionality

(2) 3. Which of the following is a filter on $\mathbb{R}$:

A. $\{A \subseteq \mathbb{R} : A \text{ is uncountable}\}$

B. $\{A \subseteq \mathbb{R} : A \text{ has cardinality } |\mathbb{R}|\}$

C. $\{A \subseteq \mathbb{R} : A \text{ is open}\}$

⇒ **D.** $\{A \subseteq \mathbb{R} : |\mathbb{R} \setminus A| < |\mathbb{R}|\}$ {easily verified, also: the other three families have disjoint members}

(2) 4. Assume ZFC. The set $H(\aleph_2)$, viewed as a structure for the language of Set Theory, does *not* satisfy which axiom:

A. Choice

B. Replacement

⇒ **C.** Power set $\{\mathcal{P}(\omega_1) \notin H(\aleph_2)\}$

D. Pairing

(2) 5. Let κ be a regular and uncountable cardinal; which statement about subsets of κ is *not* true:

A. Every stationary set is unbounded

B. Every cub set is stationary

C. Every unbounded set has cardinality κ

⇒ **D.** Every unbounded set is stationary {the set of successors is unbounded but not stationary}

(2) 6. Which of the following *ordinal* inequalities does hold:

A. $2^\omega < 3^\omega$

⇒ **B.** $\omega^2 < \omega^3$ $\{\omega^3$ is ω copies of $\omega^2\}$

C. $2 \cdot \omega < 3 \cdot \omega$

D. $2 + \omega < 3 + \omega$

More problems on the next page.

(2) 7. Which of the following *cardinal* inequalities *does not hold*:

⇒ **A.** $2^{\aleph_0} < 3^{\aleph_0}$ {This is a standard result}

B. $\aleph_0^{2013} < \aleph_{2013}$

C. $\aleph_\omega < \aleph_\omega^{\aleph_0}$

D. $2^{\aleph_0} < 2^{2^{\aleph_0}}$

(2) 8. Let $\kappa > 2^{\aleph_1}$ be regular. Which of the following is *not* provable about countable elementary substructures of $H(\kappa)$:

⇒ **A.** if S is stationary in ω_1 then for every M: if $S \in M$ then $M \cap \omega_1 \in S$ {every M contains disjoint stationary sets}

B. if S is stationary in ω_1 then there is an M such that $S \in M$ and $M \cap \omega_1 \in S$

C. if C is cub in ω_1 then for every M: if $C \in M$ then $M \cap \omega_1 \in C$

D. if C is cub in ω_1 then there is an M such that $C \in M$ and $M \cap \omega_1 \in C$

(2) 9. Which of the following partition relations is *not* provable in ZFC:

A. $\aleph_2 \to (\aleph_0, \aleph_1)^2$

B. $\aleph_2 \to (2013, \aleph_2)^2$

⇒ **C.** $\aleph_2 \to (\aleph_2, \aleph_2)^2$ {Lemma 9.4 in the book}

D. $9 \to (4, 3)^2$ {See `https://en.wikipedia.org/wiki/Ramsey's_theorem` for a table}

(2) 10. "The GCH holds below $\aleph_\eta$" means that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \eta$. Which of the following implications is *not* provable in ZFC:

A. If the GCH holds below $\aleph_\omega$ then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$ {Theorem 24.33}

⇒ **B.** If the GCH holds below $\aleph_{2013}$ then $\aleph_{2013}^{\aleph_{2013}} = \aleph_{2014}$

C. If the GCH holds below $\aleph_{\omega_2}$ then $2^{\aleph_{\omega_2}} = \aleph_{\omega_2+1}$ {Silver's theorem}

D. If the GCH holds below $\aleph_{\omega_1}$ then $\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1}$ {Theorem 5.20}

(2) 11. The weakest assumption needed to prove the statement "$|X| \leqslant |Y|$ if there is a surjection $f : Y \to X$" is

A. ZF

B. ZF plus the Countable Axiom of Choice

C. ZF plus the Principle of Dependent Choices

⇒ **D.** ZFC {The 'obvious' proof requires the Axiom of Choice, the others have a strictly countable component}

(2) 12. Which of the following statements is provable in ZFC (κ, λ and μ denote *infinite* cardinals):

A. $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+2}$

⇒ **B.** If $\mu < \kappa$ and $\mu^\lambda > \kappa$ then $\kappa^\lambda = \mu^\lambda$

C. If $\kappa < \lambda$ then $\kappa^\mu < \lambda^\mu$ {Theorem 5.20 (ii)}

D. If $\kappa < \lambda$ then $\mu^\kappa < \mu^\lambda$

More problems on the next page.

13. Recall that a set A is *finite* if there are $n \in \mathbb{N}$ and a bijection $f : n \to A$. Define A to be *D-finite* if every injective map $f : A \to A$ is surjective. In this problem we do not assume the Axiom of Choice. Prove:

(7) a. (by induction) Every $n \in \mathbb{N}$ set is D-finite (hence every finite set is D-finite).

(4) b. $\mathbb{N}$ is not D-finite.

(7) c. For a set A the following are equivalent

(1) A is *not* D-finite

(2) $|A| + 1 = |A|$, i.e., there is a bijection $f : A \to A \cup \{p\}$, where $p \notin A$

(2) $|\mathbb{N}| \leqslant |A|$, i.e., there is an injection $f : \mathbb{N} \to A$

(8) 14. Prove the first non-trivial version of Ramsey's theorem:

$$\aleph_0 \to (\aleph_0)^2_2$$

(8) 15. a. Prove the following version of the Δ-system lemma: assume $2^{\aleph_0} = \aleph_1$ and let $\langle A_\alpha : \alpha \in \omega_2 \rangle$ be a sequence of countable sets. Then there are a countable set R and a subset T of ω_2 such that $|T| = \aleph_2$ and $\{A_\alpha : \alpha \in T\}$ is a Δ-system with root R. *Hint*: Think of the set $\{\alpha \in \omega_2 : \operatorname{cf} \alpha = \aleph_1\}$ and, possibly, the pressing-down lemma.

(8) b. Let A be the set of functions from ω to 2. Prove that if $B \subseteq A$ is a Δ-system then $|B| \leqslant 2$. *Hint*: Functions are *sets* of ordered pairs.

16. For ordinal-valued functions f and g with domain ω_1 we define $f \prec g$ to mean that $\{\alpha : f(\alpha) \geqslant g(\alpha)\}$ is not stationary.

(8) a. Prove that $\prec$ is a well-founded partial order.

We denote the associated rank function by $\|\varphi\|$, so that $\|\varphi\| = \sup\{\|\psi\| + 1 : \psi \prec \varphi\}$.

(8) b. Prove $\|\varphi\| < \omega_1$ if and only if there is $\beta \in \omega_1$ such that $\{\alpha \in \omega_1 : \varphi(\alpha) = \beta\}$ is stationary, and in that case $\|\varphi\|$ is equal to the smallest such β.

(8) c. Determine the rank of the identity function on ω_1.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END

13. a. If $n \leqslant 1$ then there is only one function from n to n (the empty function if $n = 0$, and $0 \mapsto 0$ if $n = 1$).
For the inductive step let $f : n+1 \to n+1$ be injective. If $f(n) = n$ then $f \restriction n$ is an injection from n to n and hence surjective, but then f is surjective as well. If $f(n) \neq n$ then there is an $i < n$ such that $f(i) = n$, for otherwise $f \restriction n$ would be an injection from n to n that was not surjective, as $f(i) \neq f(n)$ for all $i < n$. Define $g : n+1 \to n+1$ by $g(n) = n$, $g(i) = f(n)$ and $g(j) = f(j)$ if $j < n$ and $j \neq i$. Then g is surjective by the argument above, and hence so is f.

    b. The map $n \mapsto n+1$ is injective but not surjective.

    c. (1) implies (2): let $f : A \to A$ be injective and not surjective and take $a_0 \in A \setminus f[A]$. Define $g : A \cup \{p\} \to A$ by $g(a) = f(a)$ if $a \in A$ and $g(p) = a_0$; then g is injective, hence $|A \cup \{p\}| \leqslant |A|$; since $|A| \leqslant |A \cup \{p\}|$ is clear we are done by the Cantor-Bernstein theorem.
(2) implies (3): let $f : A \cup \{p\} \to A$ be a bijection. Apply the recursion principle to obtain a map $h : \mathbb{N} \to A$ such that $h(0) = f(p)$ and $h(n+1) = f\bigl(h(n)\bigr)$ for all n. Now prove by induction that h is injective.
(3) implies (1): let $h : \mathbb{N} \to A$ be an injection and define $f : A \to A$ by $f(a) = a$ if $a \in A \setminus h[\mathbb{N}]$ and $f\bigl(h(n)\bigr) = h(n+1)$ for all n.

14. Here is one proof: let $f : [\omega]^2 \to 2$ be given. Fix a free ultrafilter $\mathcal{U}$ on ω. For each $i \in \omega$ the set $\{j : j > i\}$ is split by f into $A(i,0)$ and $A(i,1)$, where $A(i,\epsilon) = \{j > i : f(\{i,j\}) = \epsilon\}$; let ϵ_i be such that $A(i,\epsilon_i) \in \mathcal{U}$. Now take ϵ such that $A = \{i : \epsilon_i = \epsilon\} \in \mathcal{U}$. Recursively define a sequence $\langle i_n : n \in \omega\rangle$ by $i_0 = \min A$, and $i_n = \min\bigl(A \cap \bigcap_{m<n} A(i_m,\epsilon)\bigr)$ when $n > 0$. Then $f(\{i_m, i_n\}) = \epsilon$ for all m and n. For other proofs: see the book and the notes.

15. a. We may assume that $A_\alpha \subseteq \omega_2$ for all α, because $\bigcup_{\alpha\in\omega_2} A_\alpha$ has cardinality at most $\aleph_2$. Apply the hint to the function f, defined by $f(\alpha) = \sup A_\alpha \cap \alpha$. Then f is regressive on the stationary set $\{\alpha : \operatorname{cf}\alpha = \aleph_1\}$, hence constant on a stationary set S, say with value γ. By the Continuum Hypothesis the set $[\gamma]^{\leqslant\aleph_0}$ has cardinality at most $\aleph_1$, hence there is a stationary subset T of S such that $\alpha \mapsto A_\alpha \cap \alpha$ is constant, say with value R. Finally, the set $C = \{\alpha : (\forall\beta < \alpha)(\sup A_\beta < \alpha)\}$ is cub in ω_2; then $\{A_\alpha : \alpha \in T \cap C\}$ is the Δ-system that we seek.

    b. Suppose B is a non-empty Δ-system, with root R. This means that R is a function and that $\operatorname{dom} R \subseteq \omega$. In case $\operatorname{dom} R = \omega$ we have $f = R$ for all $f \in B$, so $|B| = 1$. In case there is $n \in \omega \setminus \operatorname{dom} R$ we must have $f(n) \neq g(n)$ whenever $f \neq g$ in B (as $f \cap g = R$), but there are only two possibilities for $f(n)$, so $|B| = 2$.

16. This was mostly done in class and in the homework.

    a. To see transitivity assume $\varphi \prec \psi$ and $\psi \prec \rho$. This means that $\{\alpha : \varphi(\alpha) < \psi(\alpha)\}$ and $\{\alpha : \psi(\alpha) < \rho(\alpha)\}$ both contain a cub; hence so does the intersection of this set, but $\varphi(\alpha) < \psi(\alpha) < \rho(\alpha)$ for all α in this intersection. If $\prec$ were not well-founded then we could find a sequence $\langle\varphi_n : n \in \omega\rangle$ such that $\varphi_{n+1} \prec \varphi_n$ for all n, with witnessing cub sets C_n. The set $C = \bigcap_{n\in\omega} C_n$ is cub and if $\alpha \in C$ then $\langle\varphi_n(\alpha) : n \in \omega\rangle$ would be a strictly decreasing sequence of ordinal numbers.

    b. Step 1: if $\beta \in \omega_1$ and $\{\alpha : \varphi(\alpha) \leqslant \beta\}$ is stationary then $\|\varphi\| \leqslant \beta$. The proof is by induction on β. If $\beta = 0$ then the assumption implies $\{\psi : \psi \prec \varphi\} = \emptyset$ and hence $\|\varphi\| = 0$. If $\beta > 0$ and $\psi \prec \varphi$ then $\{\alpha : \psi(\alpha) < \beta\}$ contains a cub and hence there is $\gamma < \beta$ such that $\{\alpha : \psi(\alpha) \leqslant \gamma\}$ is stationary, by induction $\|\psi\| \leqslant \gamma$. This shows that $\|\varphi\| \leqslant \sup\{\gamma + 1 : \gamma \in \beta\} = \beta$.
Step 2: if $\beta \in \omega_1$ and $\{\alpha : \varphi(\alpha) > \beta\}$ contains a cub then $\|\varphi\| > \beta$. The proof is by induction on β. If $\beta = 0$ then the zero-function, which has rank 0, is below φ and so $\|\varphi\| \geqslant 0 + 1$. If $\beta > 0$ then $\rho \prec \varphi$, where ρ is the constant function with value β. By our inductive assumption $\|\rho\| > \gamma$, whenever $\gamma < \beta$; it follows that $\|\rho\| \geqslant \beta$ and hence $\|\varphi\| \geqslant \beta + 1$.
The two steps together establish that $\|\varphi\| \leqslant \beta$ if and only $\{\alpha : \varphi(\alpha) \leqslant \beta\}$ is stationary and hence that $\|\varphi\| = \beta$ if and only if β is the smallest ordinal such that $\{\alpha : \varphi(\alpha) \leqslant \beta\}$ is stationary.

    c. Let ι denote the identity function. Note that $\varphi \prec \iota$ if and only if $\{\alpha : \varphi(\alpha) < \alpha\}$ contains a cub. So, if $\varphi \prec \iota$ then φ is regressive on a cub and hence constant on a stationary set, say with value γ. But then $\|\varphi\| \leqslant \gamma$. We see that $\|\iota\| \leqslant \sup\{\alpha + 1 : \alpha \in \omega_1\} = \omega_1$. Also, there is no $\beta \in \omega_1$ such that $\{\alpha : \iota(\alpha) \leqslant \beta\}$ is stationary, so that $\|\iota\| \geqslant \omega_1$. We conclude that $\|\iota\| = \omega_1$.

THE END