BOLZANO PARADOXIEN DES UNENDLICHEN

EINEN INBEGRIFF, DEN WIR EINEM SOLCHEN BEGRIFFE UNTERSTELLEN, BEI DEM DIE ANORDNUNG SEINER THEILE GLEICHGÜLTIG IST (AN DEM SICH ALSO NICHTS FÜR UNS WESENTLICHES ÄNDERT, WENN SICH BLOSS DIESE ÄNDERT), NENNE ICH EINE <u>MENGE</u>.

$(0,5)$ AND $(0,12)$ (INTERVALS)
HAVE THE SAME NUMBER OF ELEMENTS

EVERY x IN $(0,5)$ CORRESPONDS TO ONE (AND ONLY ONE) y IN $(0,12)$ VIA

$$12x = 5y$$

AND CONVERSELY.

CANTOR

- TRIGONOMETRIC SERIES
- UNIQUENESS THEOREMS

UNTER EINE ‚MENGE' VERSTEHEN WIR JEDE ZUSAMMENFASSUNG M VON BESTIMMTEN WOHLUNTERSCHIEDENEN OBJECTEN m UNSERER ANSCHAUUNG ~~ODER~~ ODER UNSERES DENKENS (WELCHE DIE ‚ELEMENTE' VON M GENANNT WERDEN) ZU EINEM GANZEN.

IN ZEICHEN DRÜCKEN WIR DIES SO AUS

$$M = \{m\}.$$

② ②

BOTH DEFINITIONS ARE PROBLEMATIC

- INBEGRIFF ZUSAMMENFASSUNG
  ARE BASICALLY SYNONYMS OF MENGE

- OTHER WORDS ARE QUITE
  MEANINGLESS.
  ANORDNUNG, UNSERE ANSCHAUUNG
  UNSERES DENKENS

- AND ULTIMATELY INHERENTLY
  PARADOXICAL

RUSSELL:
CONSIDER THE PROPERTY $x \notin x$
BY THE ABOVE DEFINITIONS WE
FORM THE SET R OF THOSE x-ES
THAT SATISFY $x \notin x$

NOW: $R \in R \iff R \notin R$

SOLUTION: GROUND RULES AKA AXIOMS

MUCH LIKE EUCLIDEAN GEOMETRY
STRICT RULES ON WHAT TO DO WITH
POINTS, LINES AND CIRCLES.

1 LANGUAGE: $\in$ $=$

$\wedge$ $\vee$ $\neg$ $\rightarrow$ $\leftrightarrow$

$\forall$ $\exists$

v_1 v_2 v_3 ------

2 FORMULAS:
- $x \in y$ $x = y$
- $(\varphi)\wedge(\psi)$ $(\varphi)\vee(\psi)$ $\neg(\varphi)$ $(\varphi)\rightarrow(\psi)$ $(\varphi)\leftrightarrow(\psi)$
- $(\forall x)(\varphi)$ $(\exists x)(\varphi)$

3 FREE VARIABLES - $\varphi(u_1, \ldots, u_n)$
- FREE VARIABLES AMONG $u_1, \ldots, u_n$.

$(\exists x)(x \in y)$: x BOUND
y FREE

CLASSES ENTITIES TOO BIG TO BE SETS

$R = \{x : x \notin x\}$ IS A CLASS, NOT A SET

$V = \{x : x = x\}$ IS THE CLASS OF ALL SETS,
THE <u>UNIVERSE</u>

CLASS = WHAT CANTOR WOULD HAVE US ACCEPT AS A SET

FORMULA $\varphi(x, p_1, \ldots, p_n)$ (x FREE)

$C = \{x : \varphi(x, p_1, \ldots, p_n)\}$

BUT: SETS ARE CLASSES TOO

CLASS-BUT-NOT-SET: PROPER CLASS

$C = D$ MEANS $\varphi(x, p_1 \ldots p_n) \leftrightarrow \psi(x, q_1, \ldots, q_m)$

$C \subseteq D$ $\rightarrow$

$C \cap D$ $C \cup D$ $C \setminus D$ $\bigcup C$

EXTENSIONALITY:

$(\forall u)(u \in X \leftrightarrow u \in Y) \rightarrow X = Y$

[$X = Y \rightarrow (\forall u)(u \in X \leftrightarrow u \in Y)$ IS A THEOREM FROM 1ST ORDER LOGIC]

SET EXISTENCE

$(\exists x)(\forall y)(y \notin x)$

PAIRING

$(\forall a)(\forall b)(\exists c)(\forall x)(x \in c \leftrightarrow (x = a \vee x = b))$

c IS UNIQUE

$\{a, b\}$ THAT UNIQUE c

$\langle a, b \rangle = \{\{a\}, \{a, b\}\}$

! $\langle a, b \rangle = \langle c, d \rangle \leftrightarrow (a = c \wedge b = d)$

$\langle\langle a, b \rangle, c \rangle =: \langle a, b, c \rangle$

ETC

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SEPARATION / COMPREHENSION

φ A FORMULA

$(\forall X)(\forall p)(\exists Y)(\forall u)\,(u \in Y \leftrightarrow u \in X \wedge \varphi(u,p))$

NOTE Y MAY NOT OCCUR FREE IN φ

APPLICATION

EXISTENCE OF $x \cap y$ $\quad x \setminus y$

AND $\bigcap x$

UNION

$(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow (\exists v \in x)(u \in v))$

HUMAN READABLE PROOF OF EXISTENCE OF $\bigcap x$ & ~~[illegible]~~

$\bigcap x$ IS DEFINED TO BE

$\{u : (\forall v \in x)(u \in v)\}$

OR RATHER $\{u : (\forall v)(v \in x \rightarrow u \in v)\}$

NOTE IF $x = \emptyset$ THEN $v \in x \rightarrow u \in v$ IS TRUE VACUOUSLY

SO WE WANT TO PROVE

$(*)$ $(\forall x)(x \neq \emptyset \rightarrow (\exists y)(\forall u)(u \in y \leftrightarrow (\forall v)(v \in x \rightarrow u \in v)))$

SO ASSUME $x \neq \emptyset$ AND LET $z \in x$

NOW APPLY SEPARATION TO z AND THE FORMULA $(\forall v)(v \in x \rightarrow u \in v)$

TO SEE THAT $w = \{u : u \in z \wedge (\forall v)(v \in x \rightarrow u \in v)\}$

IS A SET

NOW PROVE THAT $u \in w \leftrightarrow (\forall v)(v \in x \rightarrow u \in v)$

CHALLENGE: WRITE A FORMAL DEDUCTION OF $(*)$.