AXIOMS THUS FAR

EXTENSIONALITY, PAIRING, SEPARATION/COMPREHENSION

UNION, EXISTENCE

TODAY

POWER SET, INFINITY, REPLACEMENT,

REGULARITY

POWER SET: $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow u \subseteq x)$

$u \subseteq x$ MEANS $(\forall v)(v \in u \rightarrow v \in x)$

y IS UNIQUE ; NOTATION $\mathcal{P}(x)$.

THIS OPENS UP A LOT OF NEW POSSIBILITIES

- PRODUCT $X \times Y = \{ \langle x, y \rangle : x \in X \wedge y \in Y \}$

  $\langle x, y \rangle = \{ \{x\}, \{x, y\} \} \subseteq \mathcal{P}(X \cup Y)$

  SO $\langle x, y \rangle \in \mathcal{P}(\mathcal{P}(X \cup Y))$

- $X \times Y \times Z := (X \times Y) \times Z$
- $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ TIMES}}$ (n FOR US) !
- RELATION

  BINARY SET OF ORDERED PAIRS

  TERNARY TRIPLES

  n-ARY n-TUPLES

R BINARY $\operatorname{DOM} R \subseteq \bigcup\bigcup R$

$\operatorname{DOM} R = \{ u : (\exists v)(\langle u, v \rangle \in R) \}$

$\operatorname{RAN} R = \{ v : (\exists u)(\langle u, v \rangle \in R) \}$

$\{\{u\}, \{u, v\}\} \in R$; $\{u\}, \{u, v\} \in \bigcup R$; $u, v \in \bigcup\bigcup R$

FIELD = DOM $\cup$ RAN

FUNCTION / MAP / MAPPING:

BINARY RELATION f ST

$(\forall x)(\forall y)(\forall z)\left(\langle x,y\rangle \in f \wedge \langle x,z\rangle \in f \rightarrow y=z\right)$

f ON X IF $\operatorname{DOM} f = X$

$f: X \rightarrow Y$ ABBREVIATES
- f IS A FUNCTION
- $\operatorname{DOM} f = X$
- $\operatorname{RAN} f \subseteq Y$

Y^X OR ${}^XY$ IS THE SET OF ALL FUNCTIONS FROM X TO Y

NOTE $Y^X \subseteq \mathcal{P}(X \times Y)$

SO $Y^X \in \mathcal{P}\mathcal{P}(X \times Y)$

RESTRICTION $f \upharpoonright X = \{\langle x,y\rangle \in f : x \in X\}$
(ALSO IF $X \nsubseteq \operatorname{DOM} f$)

EXTENSION $f \subseteq g$

COMPOSITION $R \circ S = \{\langle x,z\rangle : (\exists y)(\langle x,y\rangle \in S \wedge \langle y,z\rangle \in R)\}$

f, g FUNCTION $f \circ g$ IS FUNCTION

$R^{-1} = \{\langle y,x\rangle : \langle x,y\rangle \in R\}$

IMAGE $f[X] = f''X = \{y : (\exists x \in X)(y = f(x))\}$

INFINITY.

INFINITE = NOT FINITE

FINITE = NOT INFINITE

FINITE : HAVING AN END OR A LIMIT
SUBJECT TO LIMITATIONS OR CONDITIONS
OPP TO INFINITE
(L. FINITUS -- FINIRE : TO LIMIT)

"THERE IS AN INFINITE SET"

$$(\exists S)\underbrace{(\emptyset \in S \wedge (\forall x)(x \in S \rightarrow x \cup \{x\} \in S))}_{\text{INDUCTIVE SET}}$$

NATURAL NUMBERS AND FINITE SETS : CH 2

OR WORK EXERCISES 1.3 - 1.9

OTHER APPROACHES

TARSKI-FINITE : EVERY $X \subseteq \wp(S)$ HAS
A MAXIMAL ELEMENT
$(\exists u \in X)(\forall v \in X)(u \subseteq v \rightarrow u = v)$

DEDEKIND-FINITE : EVERY INJECTIVE $f: S \to S$
IS SURJECTIVE

REPLACEMENT

φ A FORMULA $\varphi(x, y, p)$

$$(\forall x)(\forall y)(\forall z)(\varphi(x,y,p) \wedge \varphi(x,z,p) \rightarrow y = z) \longrightarrow$$

φ IS A FUNCTION

$$(\forall X)(\exists Y)(\forall y)(y \in X \leftrightarrow (\exists x \in X)(\varphi(x,y,p)))$$

THE IMAGE OF EACH SET IS A SET

NOTE THIS ALSO WORKS
IF X IS NOT ~~A~~ SUBSET OF "DOM φ"

SEE SEPARATION FOLLOWS FROM REPLACEMENT
1.14

APPLICATIONS OF REPLACEMENT:
- EXISTENCE OF UNCOUNTABLE ORDINAL NUMBERS
- BOREL DETERMINACY

EXISTENCE OF $X \times Y$.

FIX $x \in X$
APPLY REP. TO
Y AND $\varphi(y, z, x)$:
$z = \langle x, y \rangle$
GIVES
$Z = \{ \langle x, y \rangle : y \in Y \}$

APPLY REP TO
X AND $\psi(x, Z, Y)$:
$Z = \{ \langle x, y \rangle : y \in Y \}$
GIVES
$\{ \{ \langle x, y \rangle : y \in Y \} : x \in X \}$

NOW TAKE UNION TO GET $X \times Y$.

USEFUL EXERCISES

1.3 — 1.9 DO $\underline{1.3}/1.4$

1.10 $\longrightarrow$ 1.13

AXIOM OF REGULARITY

$(\forall S)(S \neq \emptyset \rightarrow (\exists x \in S)(x \cap S = \emptyset))$

NOTE WE SAY x IS $\in$-MINIMAL IN S.

CONSEQUENCE:
- NO x WITH $x = \{x\}$
- NO $x_0, x_1, \ldots, x_n$ WITH $x_0 \in x_1 \in x_2 \in \cdots \in x_n \in x_0$
- NO SEQUENCE $\langle x_n : n \in \omega \rangle$ WITH $x_0 \ni x_1 \ni x_2 \ni \cdots \ni x_n \ni x_{n+1} \ni \cdots$

IN PRACTICE REGULARITY IS NOT ESSENTIAL.
IN THEORY IT MAKES WORKING WITH MODELS FOR SET THEORY A LOT EASIER

ALSO, IT SIMPLIFIES OUR PICTURE OF THE SET-THEORETIC UNIVERSE V.

HOMEWORK
$\underline{1.5 - 1.9}$