ORDINAL NUMBERS

- PARTIAL ORDER(ING)

 - $p \not< p$
 - $p < q \wedge q < r \rightarrow p < r.$

LINEAR $\quad p<q$ OR $p=q$ OR $p>q.$

$p \leq q \quad p<q$ OR $p=q$

- A HOST OF DEFINITIONS

$X \subseteq P \quad a \in P$

a MAXIMAL IN X $\quad a \in X : \wedge \forall x \in X \; a \not< x$
a MINIMAL IN X $\quad a \in X \wedge \forall x \in X \; x \not< a$

a MAXIMUM/LARGEST ELEMENT/--- OF X: $a \in X \wedge \forall x \in X \quad x \leq a$
a MINIMUM/LEAST ELEMENT/--- OF X: $a \in X \wedge \forall x \in X \quad a \leq x$

a UPPER BOUND OF X $\quad \forall x \in X \; x \leq a$
a LOWER BOUND OF X $\quad \forall x \in X \; a \leq x$

a SUPREMUM/LEAST UPPER BOUND OF X
a INFIMUM/GREATEST LOWER BOUND OF X
} SELF EXPLANATORY

$\sup X \quad \max X \quad \min X \quad \inf X$

(LINEAR ORDER $\quad$ MAXIM<u>AL</u> IS MAXIM<u>UM</u>
MINIM<u>AL</u> IS MINIM<u>UM</u>)

- $f : (P, <_p) \longrightarrow (Q, <_q)$
 ORDER-PRESERVING IF $x<y$ IMPLIES $f(x)<f(y)$
 (LINEAR ORDERS: ALSO <u>INCREASING</u>.)

ISOMORPHISM : BIJECTION + f AND f^{-1} ORDER-PRESERVING.

2.

WELL-ORDERING

A WELL-ORDER(ING) IS A LINEAR ORDER(ING) IN WHICH EVERY NONEMPTY SUBSET HAS A MINIMUM.

(REALLY, REALLY IMPORTANT!)

2.4 $\langle W, <\rangle$ WELL-ORDERED (NONEMPTY!)
$f: W \to W$ INCREASING
THEN $\forall x \; f(x) \geq x$

PROOF: BY INDUCTION

LET $A = \{x \in W : f(x) \geq x\}$

- $\min W \in A$ CERTAINLY
  $(\forall x \in W)(\min W \leq x)$ IN PARTICULAR
  $\min W \leq f(\min W)$

- SUPPOSE $(\forall y)(y < x \to y \in A)$
  WE SHOW $x \in A$.
  NOTE $(\forall y < x)(y \underset{y \in A}{\leq} f(y) \underset{\text{INCREASING}}{<} f(x))$

  SO $(\forall y < x)(y < f(x))$
  SO OF $f(x) < x$ OR $f(x) = x$ OR $f(x) > x$
  THE FIRST IS FALSE AND SO $f(x) \geq x$.
  THAT IS, $x \in A$.

- $A = W$
  FOR IF $W \setminus A \neq \varphi$ LET $x = \min W \setminus A$
  THEN $(\forall y < x)(y \in A)$
  BUT THEN $x \in A$ CONTRADICTION.

2.5 WELL-ORDERS ARE RIGID:
THE ONLY AUTOMORPHISM IS THE IDENTITY

2.6. THERE IS AT MOST ONE ISOMORPHISM BETWEEN WELL-ORDERED SETS.

2.7. NO WELL-ORDERED SET IS ISOMORPHIC TO AN INITIAL SEGMENT OF ITSELF

INITIAL SEGMENT : A SET OF THE FORM: $\{x \in W : x < u\}$ FOR SOME $u \in W$.

2.8 COMPARABILITY THEOREM.

FOR WELL-ORDERED SETS W_1 AND W_2 EXACTLY ONE OF THE FOLLOWING HOLDS

- W_1 AND W_2 ARE ISOMORPHIC
- W_1 IS ISOMORPHIC TO AN INITIAL SEGMENT OF W_2
- W_2 IS ISOMORPHIC TO AN INITIAL SEGMENT OF W_1

PROOF

$$f = \{\langle x, y\rangle \in W_1 \times W_2 : \underbrace{\{u \in W_1 : u < x\}}_{W_1(x)} \simeq \underbrace{\{v \in W_2 : v < y\}}_{W_2(y)}\}$$

- IS A FUNCTION.

  IF $\langle x, y_1\rangle, \langle x, y_2\rangle \in f$

  THEN $W_2(y_1) \simeq W_2(y_2)$ SO, BY 2.7, NEITHER IS AN INITIAL SEGMENT OF THE OTHER, SO $y_1 = y_2$

- IF $h : W_1(x) \longrightarrow W_2(y)$ IS AN ISOMORPHISM

  THEN: IF $x' < x$ THEN $h \restriction W_1(x')$ IS AN ISOMORPHISM BETWEEN $W_1(x')$ AND $W_2(h(x'))$

VERIFY THIS CAREFULLY (AS AN EXERCISE).

- SO f IS ORDER-PRESERVING

AND, ALSO, IF $x \in \operatorname{dom} f$ AND $x' < x$ THEN $x' \in \operatorname{dom} f$.

IF $y \in \operatorname{ran} f$ AND $y' < y$ THEN $y' \in \operatorname{ran} f$

④

NOW

$\mathrm{dom}\, f = W_1 \quad \mathrm{ran}\, f = W_2$: ISOMORPHIC

$\mathrm{dom}\, f = W_1 \quad \mathrm{ran}\, f \neq W_2$: $W_1 \cong$ INIT'L SEGMENT OF W_2

$\mathrm{dom}\, f \neq W_1 \quad \mathrm{ran}\, f = W_2$ $W_2 \cong$ INITIL SEGMENT OF W_1

$\mathrm{dom}\, f \neq W_1 \quad \mathrm{ran}\, f \neq W_2$

DOES NOT OCCUR:

LET $x = \min W_1 \setminus \mathrm{dom}\, f$

$y = \min W_2 \setminus \mathrm{ran}\, f$

THEN f ITSELF WOULD WITNESS THAT $\langle x, y\rangle \in f$.

$W_1 \cong W_2$ IS EXPRESSED AS

W_1 AND W_2 HAVE THE SAME ORDER TYPE

ORDINAL NUMBER:

WE WANT ORDINAL NUMBERS (ORDINALS) TO BE WELL-ORDERED BY $\in$

- EQUAL TO THEIR INITIAL SEGMENTS, I.E.,

$\alpha = \{\beta : \beta < \alpha\}$

FOR THIS $\in$ MUST BE <u>TRANSITIVE</u> ON THESE SETS:

A SET T IS <u>TRANSITIVE</u> IF

$(\forall x \in T)(x \subseteq T)$

I.E. $(\forall x \in T)(\forall y \in x)(y \in T)$

(OR $T \subseteq P(T)$ OR $\bigcup T \subseteq T$)

AN ORDINAL (ORDINAL NUMBER) IS A <u>TRANSITIVE</u> SET WELL-ORDERED BY $\in$

- LOWER CASE GREEK: ORDINALS
- <u>ORD</u>: CLASS OF ORDINALS
- $\alpha < \beta$ MEANS $\alpha \in \beta$.

EXAMPLES

$\emptyset$, $\{\emptyset\}$, $\{\emptyset,\{\emptyset\}\}$, $\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$, ...
" " " "
0 1 2 3 .....

(THOSE THINGS FROM EXERCISES 1.3 – 1.9

PROPERTIES

- α AN ORDINAL AND $\beta \in \alpha$ $\rightarrow$ β IS AN ORDINAL
- α AND β DISTINCT ORDINALS AND $\alpha \subseteq \beta$ THEN $\alpha \in \beta$
  (TAKE $\gamma = \min \beta \setminus \alpha$
  THEN $\alpha = \{\xi \in \beta : \xi \in \gamma\} = \gamma$
  BY TRANSITIVITY).
- α AND β ORDINALS THEN $\alpha \subseteq \beta$ OR $\beta \subseteq \alpha$.
  NOTE $\alpha \cap \beta$ IS AN ORDINAL, CALL IT γ
  THEN $\gamma \subseteq \alpha$ AND $\gamma \subseteq \beta$
  SO IF $\gamma \neq \alpha$ AND $\gamma \neq \beta$ THEN $\gamma \in \alpha \cap \beta = \gamma$
  CONTRADICTION BECAUSE <u>BY DEFINITION</u>
  $\underline{\gamma \notin \gamma}$
- $<$ IS A LINEAR ORDER OF <u>ORD</u>
- $\alpha = \{\beta : \beta < \alpha\}$ (NOT A TAUTOLOGY:
  α IS THE SET OF <u>ORDINALS</u>
  THAT BELONG TO α)
- C A CLASS OF ORDINALS: $\bigcap C$ IS AN ORDINAL
  $\bigcap C = \min C$
- X A <u>SET</u> OF ORDINALS:
  $\bigcup X$ IS AN ORDINAL
  $\bigcup X = \sup X$
- $\alpha \cup \{\alpha\}$ IS AN ORDINAL
  IF $\alpha < \beta$ THEN $\alpha \cup \{\alpha\} \subseteq \beta$

  <u>$\alpha + 1$</u>

6

EVERY WELL-ORDERED SET IS ISOMORPHIC TO A UNIQUE ORDINAL.

- UNIQUE FROM ~~AN~~ RIGIDITY
- LET $\varphi(x, \alpha, W)$ MEAN
  $\langle W(x), < \rangle$ IS ISOMORPHIC TO $\langle \alpha, \in \rangle$
  $\big((\exists f)(f$ IS A FUNCTION, $\operatorname{dom} f = W(x)$, $\operatorname{ran} f = \alpha$,
  $u < v < x \rightarrow f(u) < f(v)\big)$
- SO $\forall x \in W \; \forall \alpha \; \forall \beta \; \varphi(x, \alpha, W) \wedge \varphi(x, \beta, W) \rightarrow \alpha = \beta)$
  SO $Y = \{\alpha : (\exists x \in W)\, \varphi(x, \alpha, W)\}$ EXISTS
- $F = \{\langle x, \alpha \rangle \in W \times Y : \varphi(x, \alpha, W)\}$
  IS A SET AND A FUNCTION
  IT IS INCREASING
  AND IT IS AN ISOMORPHISM BETWEEN W AND Y
  Y IS AN ORDINAL.

SUCCESSOR ORDINAL : OF THE FORM $\alpha + 1$ (OR $\alpha \cup \{\alpha\}$)

LIMIT ORDINAL : NOT SUCCESSOR

NOTE α IS A LIMIT $\leftrightarrow$ $\alpha = \bigcup \alpha = \sup \alpha$

$(\,! \sup(\alpha \cup \{\alpha\}) = \alpha < \alpha \cup \{\alpha\} \,!)$

THE SMALLEST INDUCTIVE SET IS A LIMIT ORDINAL WE CALL IT ω (ALSO $\mathbb{N}$).

FINITE AS LAST TIME
INFINITE NOT FINITE.