PROPERTIES OF ORDINALS

- IF α IS AN ORDINAL AND $\beta \in \alpha$ THEN β IS AN ORDINAL
- IF α AND β ARE ORDINALS AND $\beta \subset \alpha$ (PROPER SUBSET)

THEN $\beta \in \alpha$.

NOTE: β IS TRANSITIVE HENCE IT IS AN INITIAL SEGMENT OF α:

IF $\delta \in \gamma \in \beta$ THEN $\delta \in \alpha$

LET $\gamma = \min \alpha \setminus \beta$ THEN $\gamma = \beta$

- ALL ORDINALS ARE COMPARABLE: $\alpha \leq \beta$ OR $\beta \leq \alpha$
- C A CLASS OF ORDINALS: $\bigcap C$ IS AN ORDINAL AND $\bigcap C = \min C$.
- X A SET OF ORDINALS: $\bigcup X$ IS AN ORDINAL; $\bigcup X = \sup X$
- IF α IS AN ORDINAL THEN SO IS $\alpha \cup \{\alpha\}$ AND IF $\alpha < \beta$ THEN $\alpha \cup \{\alpha\} \leq \beta$
- α IS A SUCCESSOR ORDINAL IF $\alpha = \beta \cup \{\beta\}$ FOR SOME β. WE WRITE $\beta + 1 = \beta \cup \{\beta\}$.
- IF α IS NOT A SUCCESSOR THEN IT IS A LIMIT ORDINAL
- NOTE - α IS A LIMIT IFF $\alpha = \bigcup \alpha$
  - $\sup(\alpha \cup \{\alpha\}) = \alpha$ (!)
- $\mathbb{N}$ IS AN ORDINAL THE SMALLEST LIMIT ORDINAL ALSO DENOTED ω

NATURAL NUMBERS: THE ORDINALS BELOW ω

INDUCTION (TRANSFINITE)

IF C IS A CLASS OF ORDINALS SUCH THAT

- $0 \in C$
- IF $\alpha \in C$ THEN $\alpha + 1 \in C$
- IF α IS A LIMIT (NON ZERO) AND $(\forall \beta < \alpha)(\beta \in C)$ THEN $\alpha \in C$

THEN C IS THE CLASS OF ALL ORDINALS

②

SEQUENCES:

FUNCTIONS WITH ORDINALS AS DOMAINS

FINITE SEQUENCE: DOMAIN $\in \mathbb{N}$

i.e. $s : n \to X$

SEQUENCE DOMAIN $= \mathbb{N}$

i.e. $s : \mathbb{N} \to X$

TRANSFINITE SEQUENCE DOMAIN $> \omega$

EG $\langle a_\xi : \xi < \alpha \rangle$

$S = \langle a_\xi : \xi < \alpha \rangle$

$s^\frown x$ OR $s * x$ OR $s x$ IS

$s \cup \{\langle \alpha, x \rangle\}$

TRANSFINITE RECURSION

LET G BE A FUNCTION, DEFINED ON V

THERE IS A UNIQUE FUNCTION F,
DEFINED ON $\underline{\text{ORD}}$ SUCH THAT

$$F(\alpha) = G(F \restriction \alpha)$$

FOR ALL α.

PROOF

$$F(\alpha) = x \text{ IFF } (\exists s)(\text{dom}(s) = \alpha \wedge (\forall \xi \in \alpha)(s(\xi) = G(s \restriction \xi)) \wedge x = G(s))$$

PROVE BY INDUCTION THAT SUCH AN S IS UNIQUE (AND HENCE SO IS x), IF IT EXISTS.

PROVE BY INDUCTION THAT SUCH AN S EXISTS FOR EACH α

PROVE BY INDUCTION THAT F IS UNIQUE.

LIMITS: IF $\langle \gamma_\xi : \xi < \alpha \rangle$ IS NONDECREASING
(I.E. $\xi < \eta \to \gamma_\xi \le \gamma_\eta$)
THEN $\lim_{\xi \to \alpha} \gamma_\xi = \sup_{\xi < \alpha} \gamma_\xi$

NORMAL SEQUENCE: $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$ FOR ALL LIMIT α.
AND $\underline{\text{INCREASING}}$

ARITHMETIC

ADDITION $\alpha + 0 = \alpha$

$\alpha + (\beta+1) = (\alpha+\beta)+1$

$\alpha + \beta = \lim_{\xi \to \beta} \alpha+\xi$ (β LIMIT)

MULTIPLICATION $\alpha \cdot 0 = 0$

$\alpha \cdot (\beta+1) = \alpha \cdot \beta + \alpha$

$\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ (β LIMIT)

EXPONENTIATION $\alpha^0 = 1$

$\alpha^{\beta+1} = \alpha^{\beta} \cdot \alpha$

$\alpha^{\beta} = \lim_{\xi \to \beta} \alpha^{\xi}$ (β LIMIT)

ALSO VIA WELL-ORDERED SETS

SUM: ORDERED UNION

$\alpha \times \{0\} \cup \beta \times \{1\}$

PRODUCT $\alpha \times \beta$ LAST COORDINATE FIRST

EXPONENTS $F(\beta, \alpha)$ ...

ASSOCIATIVE: YES
COMMUTATIVE: NO
DISTRIBUTIVE: FROM THE RIGHT
INCREASING: ON THE RIGHT
SOLUTIONS: ON THE RIGHT

γ AND $\alpha \geq 0$ GIVEN: $\gamma = \alpha \cdot \beta + \delta$ HAS UNIQUE β, δ AS SOLUTION.

NORMAL FORM.

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_n} \cdot k_n$$

$n \geq 1$, $\alpha \geq \beta_1 > \beta_2 > \cdots > \beta_n$

$k_1, k_2, \ldots, k_n \in \mathbb{N} \setminus \{0\}$

ε-NUMBER: $\alpha = \omega^{\alpha}$

④

WELL-FOUNDED RELATIONS

E IS WELL-FOUNDED ON P

MEANS

FOR EVERY NONEMPTY $X \subseteq P$
THERE IS $a \in X$ SUCH THAT
$x \, E \, a$
FOR __NO__ $x \in X$.

i.e., EVERY NONEMPTY $X \subseteq P$ HAS AN E-MINIMAL ELEMENT.

RANK: $\rho : P \to \mathrm{ORD}$

$$\rho(x) = \sup\{\rho(y)+1 : y \, E \, x\}$$

$P_0 = \emptyset \qquad P_{\alpha+1} = \{x \in P : (\forall y)(y \, E \, x \to y \in P_\alpha)\}$

$P_\alpha = \bigcup_{\xi<\alpha} P_\xi \quad \alpha$ LIMIT

USE $\varphi(x,\alpha) : x \in P \wedge x \in P_{\alpha+1} \wedge x \notin P_\alpha$
AND REPLACEMENT TO SEE THAT $P_{\theta+1} = P_\theta$ FOR SOME θ

CLAIM $P = P_\theta$

THE RANK FUNCTION IS UNIQUE.

CARDINAL NUMBERS.

SAME CARDINALITY:

$|X| = |Y|$ MEANS THERE IS A BIJECTION $f : X \to Y$

SAME CARDINALITY

FINITE: $|X| = |n|$ FOR SOME $n \in \mathbb{N}$
WE WRITE $n = |n|$
X HAS n ELEMENTS

$|X| \leq |Y|$: INJECTION $f : X \to Y$

$|X| < |Y|$: $|X| \leq |Y|$ AND $|X| \neq |Y|$