CARDINAL NUMBERS

$|X| = |Y|$ MEANS: THERE IS A BIJECTION FROM X TO Y

NOTE: $|X|$ HAS, AS YET, NO MEANING.

IF X IS FINITE THEN $|X|$ GETS A MEANING: THE UNIQUE $n \in \mathbb{N}$ SUCH THAT $|X| = |n|$.

So $|n| = n$!

$|X| \leq |Y|$ MEANS: THERE IS AN INJECTION FROM X TO Y.

$|X| < |Y|$ MEANS $|X| \leq |Y|$ <u>AND</u> $|X| \neq |Y|$.

(1) CANTOR: $|X| < |\mathcal{P}(X)|$ (FORMULATED EXPLICITLY BY RUSSELL)

- $x \longmapsto \{x\}$ IS INJECTIVE, SO $|X| \leq |\mathcal{P}(X)|$
- LET $f : X \to \mathcal{P}(X)$ BE ARBITRARY

  PUT $R = \{x \in X : x \notin f(x)\}$

  IF $x \in X \setminus R$ THEN $x \in f(x) \setminus R$

  IF $x \in R$ THEN $x \in R \setminus f(x)$

  $\therefore$ FOR ALL x : $f(x) \neq R$

  SO f IS NOT SURJECTIVE.

(2) DEDEKIND - CANTOR - SCHROEDER - BERNSTEIN

IF $|A| \leq |B|$ AND $|B| \leq |A|$ THEN $|A| = |B|$.

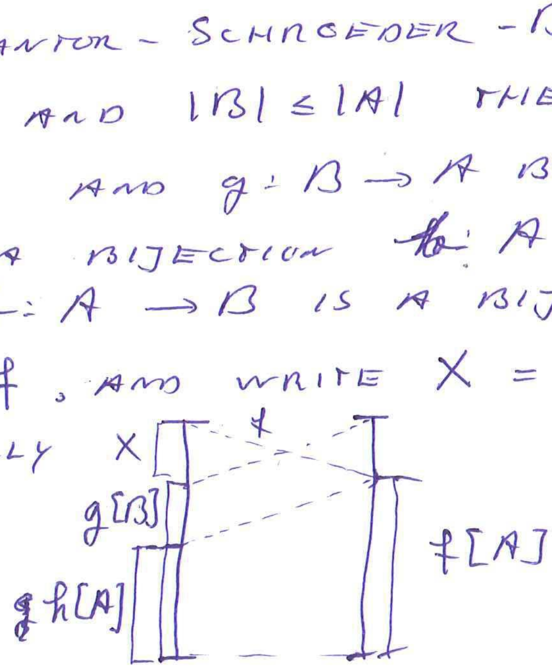
LET $f : A \to B$ AND $g : B \to A$ BE INJECTIVE.

WE CREATE A BIJECTION $h : A \to g[B]$;

THEN $g^{-1} \circ h : A \to B$ IS A BIJECTION TOO

LET $h = g \circ f$, AND WRITE $X = A \setminus g[B]$

SCHEMATICALLY



(2)

LET $\mathcal{X} = \{ Y \subseteq A : X \subseteq Y \wedge h[Y] \subseteq Y \}$

- $A \in \mathcal{X}$ FOR $X \subseteq A$ AND $h[A] \subseteq A$
- IF $\mathcal{X}' \subseteq \mathcal{X}$ THEN $\bigcap \mathcal{X}' \in \mathcal{X}$

  LET $P = \bigcap \mathcal{X}'$

  THEN. $X \subseteq P$ BECAUSE $X \subseteq Y$ FOR $Y \in \mathcal{X}'$

  . $h[P] \subseteq h[Y] \subseteq Y$ FOR ALL $Y \in \mathcal{X}'$

  SO $h[P] \subseteq \bigcap \mathcal{X}' = P$.

- LET $C = \bigcap \mathcal{X}$

  THEN C IS THE SMALLEST MEMBER OF $\mathcal{X}$.

- DEFINE $b : A \to A$ BY

$$b(x) = \begin{cases} h(x) & \text{IF } x \in C \\ x & \text{IF } x \notin C \end{cases}$$

- b IS INJECTIVE

  LET $x \neq y$

  IF $x, y \notin C$ THEN $b(x) = x \neq y = b(y)$

  IF $x, y \in C$ THEN $b(x) = h(x) \neq h(y) = b(y)$

  IF $x \in C$ AND $y \notin C$ THEN $b(x) \in C$

  SO $b(x) \neq y = b(y)$

- $b[A] = g[B]$

  LET $z \in g[B]$

  IF $z \notin C$ THEN $z = b(z)$

  IF $z \in C$ THEN CONSIDER $D = C \setminus \{z\}$

  NOTE. $X \subseteq D$ BECAUSE $z \in g[B]$

  AND $D \subset C$ (PROPER SUBSET)

  THEREFORE $h[D] \not\subseteq D$.

  NOW $h[D] \subseteq h[C] \subseteq C$

  SO WE <u>MUST</u> HAVE $z \in h[D]$

  BUT THEN ALSO $z \in h[C]$.

  THIS SHOWS $g[B] \subseteq b[A]$.

  BUT CLEARLY $b[A] \subseteq h[A] \cup g[B]$

  $= g[B]$.

CARDINAL ARITHMETIC.

~~WE WRITE κ, λ, μ FOR CARDINALITIES NUMBERS.~~

$$|A| + |B| = |A \times \{0\} \cup B \times \{1\}|$$
$$|A| \cdot |B| = |A \times B|$$
$$|A|^{|B|} = |A^B|$$

NOTE $|P(A)| = 2^{|A|}$

$X \longleftrightarrow \chi_X$ (CHARACTERISTIC FUNCTIONS).

EASY FORMULAS:

+ AND $\cdot$ ARE ASSOCIATIVE, COMMUTATIVE AND DISTRIBUTIVE ($\cdot$ OVER +).

$$\left.\begin{aligned} (\kappa \cdot \lambda)^\mu &= \kappa^\mu \cdot \lambda^\mu \\ \kappa^{\lambda+\mu} &= \kappa^\lambda \cdot \kappa^\mu \\ (\kappa^\lambda)^\mu &= \kappa^{\lambda\cdot\mu} \end{aligned}\right\} \text{OBVIOUS BIJECTIONS BETWEEN REPRESENTING SETS.}$$

$$\left.\begin{aligned} \kappa \le \lambda &\to \kappa^\mu \le \lambda^\mu \\ 0 < \lambda \le \mu &\to \kappa^\lambda \le \kappa^\mu \end{aligned}\right\} \text{OBVIOUS INJECTIONS.}$$

AN ORDINAL ~~#~~, α, IS A CARDINAL (NUMBER) IF $|\beta| \neq |\alpha|$ FOR ALL $\beta < \alpha$.

- EVERY $m \in \mathbb{N}$ IS A CARDINAL NUMBER
- ω IS A CARDINAL; THE FIRST/LEAST/SMALLEST INFINITE CARDINAL
- IF W IS WELL-ORDERED THEN

$$|W| = \min\{\alpha : |W| = |\alpha|\}$$

THIS LOOKS CIRCULAR BUT WE CAN REPLACE IT BY

$$|W| = \min\{\alpha : \text{THERE IS A BIJECTION } b : W \to \alpha\}$$

④

LET X BE A SET

$\mathcal{W} = \{ R \subseteq X^2 :$ R WELL-ORDERS FIELD$(R)\}$

IS A SET

SO $\{ \alpha : (\exists R \in \mathcal{W})(R$ ISOMORPHIC TO $\alpha)\}$

IS A SET OF ORDINALS

IT IS OF COURSE, TRANSITIVE HENCE

AN ORDINAL ITSELF, CALL IT $h(X)$.

NOTE $h(X)$ IS THE SMALLEST α SUCH THAT NO INJECTION $f : \alpha \to X$ EXISTS.

IT IS A CARDINAL

- IF α IS AN ORDINAL THEN $h(\alpha)$ IS THE SMALLEST CARDINAL ABOVE α.

- IF X IS A SET OF CARDINALS THEN $\sup X$ IS A CARDINAL.

BY RECURSION

$\aleph_0 = \omega_0 = \omega$

$\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+ \quad (= h(\aleph_\alpha))$

$\aleph_\alpha = \omega_\alpha = \sup_{\beta < \alpha} \omega_\beta \qquad \alpha$ LIMIT

IF $|X| = \omega$ THEN X IS COUNTABLE

$\aleph_{\alpha+1}$ IS A SUCCESSOR CARDINAL

OTHERWISE $\aleph_\alpha$ IS A LIMIT CARDINAL

3.5 $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ [HESSENBERG]

WE WELL-ORDER ORD × ORD:

$\langle \alpha, \beta \rangle < \langle \gamma, \delta \rangle$ IFF $\max\{\alpha,\beta\} < \max\{\gamma,\delta\}$

OR $\max\{\alpha,\beta\} = \max\{\gamma,\delta\}$ AND $\alpha < \gamma$

OR $\max\{\alpha,\beta\} = \max\{\gamma,\delta\}$ AND $\alpha = \gamma$ AND $\beta < \delta$.

THIS IS A WELL-ORDER OF ORD x ORD

- $\alpha \times \alpha = \{ \langle \gamma, \delta \rangle : \langle \gamma, \delta \rangle < \langle 0, \alpha \rangle \}$
- $\Gamma(\alpha, \beta) =$ ORDER TYPE OF $\{ \langle \gamma, \delta \rangle : \langle \gamma, \delta \rangle < \langle \alpha, \beta \rangle \}$

Γ IS INCREASING FROM ORD x ORD TO ORD

$\gamma(\alpha) = \Gamma[\alpha \times \alpha] = \Gamma(0, \alpha)$ IS (ALSO) INCREASING

SO $\gamma(\alpha) \geq \alpha$ FOR ALL α.

NOTE IF α IS A LIMIT THEN

$$\alpha \times \alpha = \bigcup_{\beta < \alpha} \beta \times \beta$$

SO THAT $\gamma(\alpha) = \lim_{\beta \to \alpha} \gamma(\beta)$.

WE SEE THAT γ IS A NORMAL FUNCTION,

SO $\gamma(\alpha) = \alpha$ FOR MANY ORDINALS α.

CLAIM $\Gamma[\kappa \times \kappa] = \kappa$ FOR EVERY CARDINAL NUMBER κ (INFINITE)

INDUCTION

$\Gamma[\omega_0 \times \omega_0] = \omega_0$: EACH $\langle m, n \rangle$ HAS FINITELY MANY PREDECESSORS.

$\Gamma[\omega_{\alpha+1} \times \omega_{\alpha+1}] = \omega_{\alpha+1}$: GIVEN THAT $\Gamma[\omega_\alpha \times \omega_\alpha] = \omega_\alpha$.

FOR EVERY $\delta < \omega_{\alpha+1}$ WE HAVE

$|\delta| \leq \omega_\alpha$ HENCE $|\delta \times \delta| \leq \omega_\alpha \cdot \omega_\alpha = \omega_\alpha$

HENCE IF $\gamma, \delta < \omega_{\alpha+1}$ THEN

$\Gamma(\gamma, \delta)$ HAS CARDINALITY AT MOST ω_α

THUS $\sup_{\gamma, \delta < \omega_{\alpha+1}} \Gamma(\gamma, \delta) \leq \omega_{\alpha+1}$

THAT IS $\gamma(\omega_{\alpha+1}) \leq \omega_{\alpha+1}$ AND SO $\gamma(\omega_{\alpha+1}) = \omega_{\alpha+1}$.

(6)

COFINALITY

LET α BE A LIMIT

$\langle \alpha_\xi : \xi < \beta \rangle$ (INCREASING)

IS COFINAL IN α IF $\lim_{\xi \to \beta} \alpha_\xi = \alpha$.

$A \subseteq \alpha$ IS COFINAL IF $\alpha = \sup A$.

$\mathrm{CF}\,\alpha$ = SMALLEST β SUCH THAT SOME $\langle a_\xi : \xi < \beta \rangle$ IS COFINAL IN α.

- $\mathrm{CF}\,\alpha$ IS A LIMIT ORDINAL
- $\mathrm{CF}\,\alpha \leq \alpha$
- $\mathrm{CF}(\omega + \omega) = \omega$
- $\mathrm{CF}(\mathrm{CF}\,\alpha) = \mathrm{CF}\,\alpha$

  $\leq$ IS ALWAY TRUE

  $\geq$ IF $\langle a_\xi : \xi < \beta \rangle$ IS COFINAL IN α AND $\langle \beta_\eta : \eta < \gamma \rangle$ IS COFINAL IN β THEN $\langle \alpha_{\beta_\eta} : \eta < \gamma \rangle$ IS COFINAL IN α.

- $\alpha > 0$ A LIMIT $A \subseteq \alpha$ COFINAL $\rightarrow$ $\mathrm{tp}\,A \geq \mathrm{CF}\,\alpha$.

  USE THE ENUMERATION OF A

- IF $\langle \beta_\xi : \xi < \gamma \rangle$ IS NON-DECREASING AND COFINAL IN α THEN $\mathrm{CF}\,\gamma = \mathrm{CF}\,\alpha$
  - $\mathrm{CF}\,\alpha \leq \mathrm{CF}\,\gamma$: USE SETS COFINAL IN γ
  - $\mathrm{CF}\,\alpha \geq \mathrm{CF}\,\gamma$ $\alpha = \lim_{\eta \to \mathrm{CF}\alpha} \alpha_\eta$

    FOR $\eta < \mathrm{CF}\,\alpha$: $\xi(\eta) = \min\{\xi : \beta_\xi > \alpha_\eta,\ \xi > \xi(\nu)\ (\nu < \eta)\}$

    THEN ~~CF~~ $\gamma = \lim_{\eta \to \mathrm{CF}\alpha} \xi(\eta)$

- $\mathrm{CF}\,\alpha$ IS A CARDINAL

  IF $\beta < \mathrm{CF}\,\alpha$ AND IF $b : \beta \to \mathrm{CF}\,\alpha$ WERE A BIJECTION THEN WE COULD SHOW $\mathrm{CF}\,\mathrm{CF}\,\alpha \leq \beta < \mathrm{CF}\,\alpha$

- $\aleph_\alpha$ IS <u>REGULAR</u> IF $\mathrm{CF}\,\omega_\alpha = \omega_\alpha$

  OTHERWISE <u>SINGULAR</u>

IF κ IS AN ALEPH THEN

- IF $X \subseteq \kappa$ AND $|X| < \mathrm{cf}\,\kappa$ THEN $\sup X < \kappa$
- IF $\lambda < \mathrm{cf}\,\kappa$ AND $f : \lambda \to \kappa$ THEN $\sup f[\lambda] < \kappa$.

$\aleph_{\alpha+\omega}$ IS ALWAYS SINGULAR

$\aleph_{\alpha+1}$ IS REGULAR – PROVIDED AC HOLDS

κ IS SINGULAR IFF THERE ~~ARE~~ $\lambda < \kappa$ AND $\{S_\xi : \xi < \lambda\}$ SUCH THAT

- $|S_\xi| < \kappa$ FOR ALL ξ
- $\kappa = \bigcup_{\xi<\lambda} S_\xi$

(THE SMALLEST SUCH IS $\mathrm{cf}\,\kappa$)

$\Rightarrow$ $\lambda = \mathrm{cf}\,\kappa$ $\quad S_\xi = \alpha_\xi \quad (\kappa = \lim_{\xi\to\lambda} \alpha_\xi)$

$\Leftarrow$ LET λ BE THE SMALLEST POSSIBLE THAT WORKS

$\beta_\xi = \cancel{tp\, S_\beta}\ tp \bigcup_{\eta<\xi} S_\eta$

$\eta < \xi \to \beta_\eta \le \beta_\xi$

MINIMALITY OF λ: $\beta_\xi < \kappa$ ALWAYS

$\beta = \lim_{\xi\to\lambda} \beta_\xi$ DEFINE $f : \kappa \to \lambda \times \beta$ BY $f(\alpha) = \langle \xi, \gamma \rangle$ WHERE

$\xi = \min\{\eta : \alpha \in S_\eta\}$

$\gamma = tp\,(S_\xi \cap \alpha)$

NOTE f IS INJECTIVE SO $|\lambda| \cdot |\beta| \ge \kappa$

BUT $\lambda < \kappa$ SO $|\beta| \ge \kappa$.

KÖNIG (SPECIAL CASE)

IF κ IS INFINITE THEN $\kappa < \kappa^{\mathrm{cf}\,\kappa}$.

$\le$: USE CONSTANT FUNCTIONS

$<$: LET $F : \kappa \to \kappa^{\mathrm{cf}\,\kappa}$ $\quad F = \langle f_\alpha : \alpha < \kappa \rangle$

TAKE $\langle \alpha_\xi : \xi < \mathrm{cf}\,\kappa \rangle$ BE COFINAL IN κ.

FOR $\xi < \kappa$ LET $f(\xi) = \min \kappa \setminus \{f_\alpha(\xi) : \alpha < \alpha_\xi\}$

NOTE - $f \ne f_\alpha$ FOR ALL α

- $f(\xi)$ EXISTS AS $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \le |\alpha_\xi| < \kappa$.