Regular cardinal: $\operatorname{cf} \kappa = \kappa$

Singular cardinal $\operatorname{cf} \kappa < \kappa$.

$\aleph_{\alpha+\omega}$ is always singular

$\aleph_{\alpha+1}$ is regular provided AC holds

(1) κ is singular iff there are $\lambda < \kappa$ and $\langle S_\xi : \xi < \lambda \rangle$ in $\mathcal{P}(\kappa)$ such that $|S_\xi| < \kappa$ for all ξ and $\kappa = \bigcup_{\xi<\lambda} S_\xi$

$\Rightarrow$ If $\langle \alpha_\xi : \xi < \operatorname{cf} \kappa \rangle$ is increasing and cofinal then $\lambda = \operatorname{cf} \kappa$ and $S_\xi = \alpha_\xi$ work.

$\Leftarrow$ Let λ be minimal with this property

For $\eta < \lambda$ let $\beta_\eta = \operatorname{tp} \bigcup_{\xi<\eta} S_\xi$

By minimality of λ we have $|\beta_\eta| < \kappa$

hence $\beta_\eta < \kappa$.

Also $\eta < \xi \to \beta_\eta \leq \beta_\xi$

Let $\beta = \lim_{\eta \to \lambda} \beta_\eta$

Define $f : \kappa \to \lambda \times \beta$

$\alpha \longmapsto \langle \eta, \gamma \rangle$ $\quad \eta = \min\{\xi : \alpha \in S_\xi\}$

$\gamma = \operatorname{tp}(S_\eta \cap \alpha)$

Then f is injective, so $|\lambda| \cdot |\beta| \geq \kappa$.

But $|\lambda| < \kappa$ so $|\beta| \geq \kappa$

and so $\beta = \kappa$.

(2) König's inequality

$$\kappa < \kappa^{\operatorname{cf} \kappa}$$

If $F : \kappa \to \kappa^{\operatorname{cf} \kappa}$ define $f(\xi) = \min \kappa \setminus \{F(\alpha)(\xi) : \alpha < \alpha_\xi\}$

$\langle \alpha_\xi : \xi < \operatorname{cf} \kappa \rangle$ incr. and cof in κ $\quad$ then $f \neq F(\alpha)$ for all α.

AXIOM OF CHOICE

IF $\emptyset \notin S$ THEN THERE IS $f: S \to \bigcup S$
SUCH THAT $f(x) \in x$ FOR $x \in S$

EXERCISE 2.5

CONVERSE : IF $A \subseteq W$ HAS NO MINIMUM
THEN $S_a = \{x \in A : x < a\} \neq \emptyset$ $(a \in A)$

HOW TO DEFINE A DECREASING SEQUENCE? RECURSION?
THAT NEEDS A <u>FUNCTION</u> TO START WITH!
IF WE HAVE $f: A \to A$ WITH
$f(a) \in S_a$ FOR ALL a.
THEN WE TAKE $a_0 \in A$ AND
RECURSIVELY SET $a_{n+1} = f(a_n)$.
THIS APPLIES AC TO GET f.

ZERMELO'S WELLORDERING THEOREM

LET X BE A SET AND $f: \mathcal{P}(X) \setminus \{\emptyset\} \to X$
A CHOICE FUNCTION
RECURSION! THERE IS $F: \Theta \to X$
SUCH THAT $F(\alpha) = f(X \setminus \{F(\xi) : \xi < \alpha\})$
IF $X \setminus \{F(\xi) : \xi < \alpha\} \neq \emptyset$

ALSO $|A| \leq |B|$ IFF THERE IS $f: B \twoheadrightarrow A$ (ONTO)

UNION OF COUNTABLY MANY CBLE SETS IS CBLE

GENERAL: $|\bigcup S| \leq |S| \cdot \sup\{|x| : x \in S\}$

EACH $\aleph_{\alpha+1}$ IS REGULAR

ZORN'S LEMMA

COUNTABLE AXIOM OF CHOICE

FOR COUNTABLE FAMILIES

DEPENDENT CHOICES:

E A BINARY RELATION ON A ST

$(\forall a \in A)(\exists b \in A)(b \, E \, a)$

THERE IS $\langle a_n : n < \omega \rangle$ WITH $a_{n+1} \, E \, a_n$ FOR ALL n.

CHARACTERIZE WELL-ORDERED / WELL-FOUNDED VIA NO DECREASING SEQUENCES

CARDINAL ARITHMETIC

- $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$

- $2 \leq \kappa \leq \lambda \rightarrow \kappa^\lambda = 2^\lambda$

  FOR $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$

- WHAT IS κ^λ IF $\lambda < \kappa$?

  CASE 1 $\kappa \leq 2^\lambda$: $\kappa^\lambda = 2^\lambda$ AS $\kappa^\lambda \leq 2^{\lambda \cdot \lambda} = 2^\lambda$

  CASE 2 $2^\lambda < \kappa$: $\kappa \leq \kappa^\lambda \leq 2^\kappa$

  $\kappa < \kappa^\lambda$ IF $\lambda \geq \mathrm{cf}\,\kappa$

- IF $|A| = \kappa$ THEN $|[A]^\lambda| = \kappa^\lambda$ $\quad([A]^\lambda = \{X \subseteq A : |X| = \lambda\})$

  - IF $f : \lambda \to A$ THEN $f \subseteq \lambda \times A$ AND $|f| = \lambda$

    SO $\kappa^\lambda \leq |[\lambda \times A]^\lambda| = |[A]^\lambda|$

  - ALSO $f \mapsto \operatorname{ran} f$ IS ONTO $[A]^\lambda$ FROM $\{f \in A^\lambda : f$ IS INJECTIVE$\}$

    SO $|[A]^\lambda| \leq |{}^\lambda A| = \kappa^\lambda$.

$$\kappa^{<\lambda} = \sup\{\kappa^{\mu} : \mu < \lambda \text{ cardinal}\}$$

$$[A]^{<\kappa} = P_\kappa(A) = \{X \subseteq A : |X| < \kappa\}$$

THEN $|P_\kappa(A)| = |A|^{<\kappa}$.

SUMS:

$$\sum_{i\in I} \kappa_i = \Big|\bigcup_{i\in I} \{i\}\times\kappa_i\Big|$$

- $\kappa_i = \kappa$ FOR ALL i: $\sum_{i\in I}\kappa_i = \kappa\cdot|I|$
- λ INFINITE $\kappa_i > 0$ FOR ALL $i<\lambda$

$$\sum_{i<\lambda}\kappa_i = \lambda\cdot\sup_{i<\lambda}\kappa_i$$

$\leq$ CLEAR $(i,\alpha)\longmapsto(i,\alpha)$
IS INJECTION FROM
$\bigcup_{i<\lambda}\{i\}\times\kappa_i$ INTO $\lambda\times(\sup_{i<\lambda}\kappa_i)$

$\geq$ $\lambda = \sum_{i<\lambda}1 \leq \sum_{i<\lambda}\kappa_i$

$\kappa_j \leq \sum_{i<\lambda}\kappa_i$ FOR ALL j

SO $\sup_{i<\lambda}\kappa_i \leq \sum_{i\in I}\kappa_i$

SO $\lambda\cdot\sup_{i<\lambda}\kappa_i = \max\{\lambda, \sup_{i<\lambda}\kappa_i\} \leq \sum_{i<\lambda}\kappa_i$

- κ IS SINGULAR $\Leftrightarrow$ $\kappa = \sum_{i<\lambda}\kappa_i$ WITH SOME $\lambda<\kappa$ AND $\kappa_i<\kappa$.

PRODUCTS

$$\prod_{i\in I}\kappa_i = \Big|\prod_{i\in I}\kappa_i\Big|$$

$\sum_{i\in I}\kappa_i \leq \prod_{i\in I}\kappa_i$ IF $\kappa_i \geq 2$ FOR ALL I

$$f: \bigcup_{i\in I}\{i\}\times\kappa_i \longrightarrow \prod_{i\in I}\kappa_i \qquad \langle i,\alpha\rangle \longmapsto f_{\langle i,\alpha\rangle} \begin{array}{l} i\mapsto\alpha \\ j\mapsto 0 \quad j\neq i \end{array} \Big|\ \alpha\neq 0$$

$$f_{\langle i,0\rangle} \begin{array}{l} i\mapsto 0 \\ j\mapsto 1 \quad j\neq i \end{array} \Big)\ \alpha = 0$$

- $\prod_{i<\lambda} \kappa_i = (\sup \kappa_i)^\lambda$ $\quad \langle \kappa_i : i<\lambda \rangle$ NON DECREASING
  $\aleph_0$

$\leq$ CLEAR $\prod_{i<\lambda} \kappa_i \subseteq (\underbrace{\sup \kappa_i}_{\kappa})^\lambda$

$\geq$ VIA $\Gamma : \lambda \times \lambda \to \lambda$

SPLIT λ INTO $\{A_i : i<\lambda\}$ EACH OF SIZE λ

SO $\sup_{i \in A_j} \kappa_i = \kappa$ FOR ALL j

$\prod_{i\in\lambda} \kappa_i = \prod_{j\in\lambda} \left(\prod_{i \in A_j} \kappa_i \right) \geq \prod_{j\in\lambda} \kappa = \kappa^\lambda$

KÖNIG AGAIN:

IF $\kappa_i < \lambda_i$ FOR ALL i

THEN $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$

$\leq$: $(i,\alpha) \longmapsto f_{i,\alpha} \in \prod_{j\in I} \lambda_i$ $\quad f_{i,\alpha}(i) = \alpha$

$f_{i,\alpha}(j) = \kappa_j \quad j \neq i$

$<$ LET $f : \bigcup_{i\in I} \{i\} \times \kappa_i \to \prod_{i\in I} \lambda_i$

NOTE $\{ (f(i,\alpha))_i : \alpha \in \kappa_i \} \subseteq \lambda_i$ AND INCLUSION IS STRICT

SO TAKE $x \in \prod_{i\in I} \lambda_i$ $\quad x_i \neq f(i,\alpha)_i \quad \alpha<\kappa_i$

$\kappa < 2^\kappa$ SPECIAL CASE: $\kappa_i = 1 \quad \lambda_i = 2$

$\kappa < \kappa^{cf\,\kappa}$ — " — $\quad \langle \kappa_i : i < cf\,\kappa \rangle$ INCR. $\sup = \kappa$

$\lambda_i = \kappa$

$cf(2^\kappa) > \kappa$

IF $\langle \kappa_i : i<\kappa \rangle$ IS SEQ IN 2^κ

THEN $\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} \kappa_i \leq (2^\kappa)^\kappa = 2^\kappa$

$cf(\kappa^\lambda) > \lambda$

DITTO