Sums and products

$$\sum_{i\in I}\kappa_i = \left|\bigcup_{i\in I}\{i\}\times\kappa_i\right|$$

Easy properties

- $\kappa_i=\kappa$ for all i : $\sum_{i\in I}\kappa_i = |I|\cdot\kappa$
- λ an infinite cardinal $\kappa_i>0$ for all $i<\lambda$:

$$\sum_{i\in\lambda}\kappa_i = \lambda\cdot\sup_{i\in\lambda}\kappa_i$$

$\leq$ $\bigcup_{i\in\lambda}\{i\}\times\kappa_i$ is a subset of $\lambda\times(\sup_{i\in\lambda}\kappa_i)$

$\geq$ $\lambda = \sum_{i\in\lambda}1 \leq \sum_{i\in\lambda}\kappa_i$

$\kappa_j\leq\sum_{i\in\lambda}\kappa_i$ for all j, so $\sup_{i\in\lambda}\kappa_i\leq\sum_{i\in I}\kappa_i$

so $\lambda\cdot\sup_{i\in\lambda}\kappa_i = \max\{\lambda,\sup_{i\in\lambda}\kappa_i\}\leq\sum_{i\in\lambda}\kappa_i$

- κ singular $\iff$ $\kappa=\sum_{i\in\lambda}\kappa_i$ for some $\lambda<\kappa$ and $\kappa_i<\kappa$.

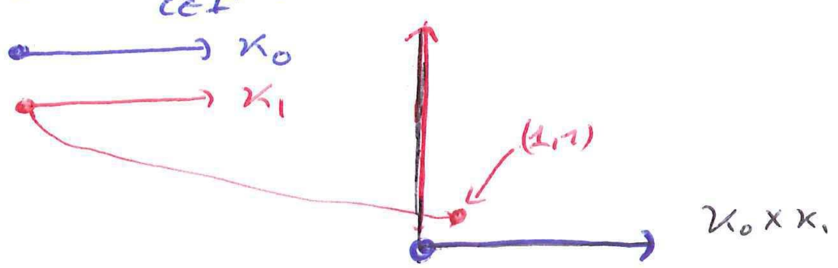
$$\underbrace{\prod_{i\in I}\kappa_i}_{\text{cardinal number}} = \Big|\underbrace{\prod_{i\in I}\kappa_i}_{\text{set}}\Big|$$

Remember $\prod_{i\in I}X_i=\{f : f$ function, dom $f=I$, $(\forall i\in I)(f(i)\in X_i)\}$

- if $\kappa_i\geq 2$ for all i

then $\sum_{i\in I}\kappa_i\leq\prod_{i\in I}\kappa_i$

$|I|=2$:



$|I|\geq 3$

$$f:\bigcup_{i\in I}\{i\}\times\kappa_i\longrightarrow\prod_{i\in I}\kappa_i$$

$\langle i,\alpha\rangle\longmapsto f(i,\alpha):\ i\mapsto\alpha,\ j\mapsto 0\ (j\neq i)$ $\Big\}\ \alpha\neq 0$

$\langle i,0\rangle\longmapsto f(i,0):\ i\mapsto 0,\ j\mapsto 1\ \ j\neq i$

injective because $|I|\geq 3$!

- $\prod_{i \in \lambda} \kappa_i = (\sup_{i\in\lambda} \kappa_i)^\lambda$ PROVIDED $\langle \kappa_i : i \in \lambda \rangle$ NON-DECREASING

$\leq$: $\prod_{i\in\lambda} \kappa_i \leq (\sup_{i\in\lambda} \kappa_i)^\lambda$

$\geq$ USE $\Gamma : \lambda \times \lambda \to \lambda$ TO WRITE

$\lambda = \bigcup_{i\in\lambda} A_i$ WITH - $i \neq j \to A_i \cap A_j = \emptyset$
- $|A_i| = \lambda$

PUT $\kappa = \sup_{i\in\lambda} \kappa_i$ SO $\kappa = \sup_{i\in A_j} \kappa_i$ FOR ALL j.

$$\prod_{i\in\lambda} \kappa_i = \prod_{j\in\lambda} \prod_{i\in A_j} \kappa_i \geq \prod_{j\in\lambda} \kappa = \kappa^\lambda.$$

KÖNIG'S INEQUALITY

IF $\kappa_i < \lambda_i$ FOR ALL i THEN

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$$

$\leq$ $\langle i, \alpha \rangle \longmapsto f(i,\alpha) : \begin{array}{l} i \mapsto \alpha \\ j \mapsto \kappa_j \quad j \neq i \end{array}$

$<$ GIVEN $f : \bigcup_{i\in I} \{i\} \times \kappa_i \to \prod_{i\in I} \lambda_i$

LOOK AT $A_i = \{ f(i,\alpha)(i) : \alpha \in \kappa_i \}$

BECAUSE $|A_i| \leq \kappa_i < \lambda_i$

WE CAN LET $g(i) = \min \lambda_i \setminus A_i$

TO GET $g \in \prod_{i\in I} \lambda_i$ WITH $g \neq f(i,\alpha)$ FOR ALL $\langle i, \alpha \rangle$.

APPLICATIONS

1. $\kappa < 2^\kappa$: $\kappa_i = 1$, $\lambda_i = 2$
2. $\kappa < \kappa^{cf\,\kappa}$: $\langle \kappa_i : i < cf\,\kappa \rangle$ INCREASING COFINAL IN κ AND $\lambda_i = \kappa$ FOR ALL i
3. $\kappa < cf(2^\kappa)$: TAKE ANY SEQ $\langle \kappa_i : i < \kappa \rangle$ IN 2^κ AND LET $\lambda_i = 2^\kappa$

   THEN $\sup_i \kappa_i \leq \sum_i \kappa_i < \prod_i 2^\kappa = 2^\kappa$.
4. MORE GENERALLY $\lambda < cf(\kappa^\lambda)$

NOTE IN PARTICULAR $\mathrm{CF}\, 2^{\aleph_0} \geq \aleph_1$

THIS IS THE <u>ONLY</u> RESTRICTION: IT IS CONSISTENT WITH ZFC TO HAVE

$$2^{\aleph_0} = \kappa$$

FOR ANY GIVEN κ WITH $\mathrm{CF}\,\kappa > \aleph_0$

CONTINUUM FUNCTION:

WHAT CAN WE SAY ABOUT $\kappa \longmapsto 2^\kappa$.

SIMPLEST POSSIBILITY

GENERALIZED CONTINUUM HYPOTHESIS:

$$(\forall \alpha)\left(2^{\aleph_\alpha} = \aleph_{\alpha+1}\right)$$

IF GCH HOLDS THEN κ^λ IS EASY:

- $\kappa \leq \lambda \longrightarrow \kappa^\lambda = \lambda^+$ $\quad (\kappa^\lambda = 2^\lambda = \lambda^+)$
- $\mathrm{CF}\,\kappa \leq \lambda < \kappa \longrightarrow \kappa^\lambda = \kappa^+$ $\quad (\kappa < \kappa^{\mathrm{CF}\,\kappa} \leq \kappa^\lambda \leq \kappa^\kappa = 2^\kappa)$
- $\lambda < \mathrm{CF}\,\kappa \longrightarrow \kappa^\lambda = \kappa$ $\quad$ AS A SET: $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$

BETH: $\beth_0 = \aleph_0$

$\beth_{\alpha+1} = 2^{\beth_\alpha}$

$\beth_\alpha = \sup_{\beta<\alpha} \beth_\beta$ $\quad$ (α LIMIT)

WITH NO SPECIAL ASSUMPTIONS:

- $\kappa < \lambda \longrightarrow 2^\kappa \leq 2^\lambda$ $\quad$ [STRICT INEQUALITY NOT PROVABLE]
- $\mathrm{CF}\, 2^\kappa > \kappa$ $\quad$ [SEE PAGE 2]
- κ A LIMIT CARDINAL: $2^\kappa = (2^{<\kappa})^{\mathrm{CF}\,\kappa}$.

  SAY $\kappa = \sum_{i \in \mathrm{CF}\,\kappa} \kappa_i$ $\quad$ ($\kappa_i < \kappa$ FOR ALL i)

  $$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \leq \prod_i 2^{<\kappa} = (2^{<\kappa})^{\mathrm{CF}\,\kappa} \leq 2^\kappa.$$

④

If κ is singular and there are $\mu < \kappa$ and λ such that
$2^{\nu} = \lambda$ if $\mu \leq \nu < \kappa$
Then $2^{\kappa} = \lambda$.

W.L.O.G. $\mu \geq cf\kappa$

So $2^{\mu} = \lambda = 2^{<\lambda}$

But then $2^{\kappa} = (2^{<\kappa})^{cf\kappa} = (2^{\mu})^{cf\kappa} = 2^{\mu} = \lambda$.

Gimel $\gimel(\kappa) = \kappa^{cf\kappa}$

The Gimel function determines the Continuum function.

If κ is a limit cardinal and
$\forall \mu < \kappa \; \exists \nu < \kappa \quad 2^{\mu} < 2^{\nu}$
Then $\lambda = 2^{<\kappa} = \lim_{\alpha \to \kappa} 2^{|\alpha|}$ has cofinality $cf\kappa$.
So $2^{\kappa} = (2^{<\kappa})^{cf\kappa} = \lambda^{cf\lambda}$

If κ is also regular then $2^{\kappa} = \kappa^{\kappa} = \kappa^{cf\kappa}$

Thus:
- κ a successor $2^{\kappa} = \gimel(\kappa)$
- κ limit and 2^{μ} constant on a tail below κ: $2^{\kappa} = 2^{<\kappa} \cdot \gimel(\kappa)$
- κ limit and 2^{μ} not constant on a tail below κ: $2^{\kappa} = \gimel(2^{<\kappa})$

Cardinal exponentiation

Hausdorff's formula
$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha}^{\aleph_\beta} \cdot \aleph_{\alpha+1}$$
Certainly if $\beta \geq \alpha$: $2^{\aleph_\beta}$ on both sides

If $\beta \leq \alpha$

Then every function $f: \aleph_\beta \to \aleph_{\alpha+1}$ is bounded

So $\{f : f: \aleph_\beta \to \aleph_{\alpha+1}\} = \bigcup_{\gamma < \omega_{\alpha+1}} \gamma^{\omega_\beta}$

- κ a limit cardinal & $\lambda \geq cf\kappa$

Then $\kappa^\lambda = (\lim_{\alpha \to \kappa} \alpha^\lambda)^{cf\kappa}$.

Say $\kappa = \sum_{i < cf\kappa} \kappa_i$ with $\kappa_i < \kappa$ for all i

$\kappa^\lambda = (\prod_{i < cf\kappa} \kappa_i)^\lambda = \prod_{i < cf\kappa} \kappa_i^\lambda \leq \prod_{i < cf\kappa} (\lim_{\alpha \to \kappa} \alpha^\lambda) = (\lim_{\alpha \to \kappa} \alpha^\lambda)^{cf\kappa}$

$\leq (\kappa^\lambda)^{cf\kappa} = \kappa^\lambda$.

Summary:

(i) $\kappa \leq \lambda \to \kappa^\lambda = 2^\lambda$

(ii) If $\mu^\lambda \geq \kappa$ for some $\mu < \kappa$ then $\kappa^\lambda = \mu^\lambda$

(iii) If $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$

then

(a) if $\lambda < cf\kappa$ then $\kappa^\lambda = \kappa$

(b) if $\lambda \geq cf\kappa$ then $\kappa^\lambda = \kappa^{cf\kappa}$

(i) old

(ii) $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$

(iii) κ successor : Hausdorff's formula

$(\kappa^+)^\lambda = \kappa^\lambda . \kappa^+$ (case (a))

κ limit $\lim_{\alpha \to \kappa} \alpha^\lambda = \kappa$

$\lambda < cf\kappa$: $\kappa^\lambda = \bigcup_{\alpha < \kappa} \alpha^\lambda$ so $\kappa^\lambda = \sup_{\alpha < \kappa} |\alpha|^\lambda = \kappa$

$cf\kappa \leq \lambda$: $\kappa^\lambda = (\lim_{\alpha \to \kappa} \alpha^\lambda)^{cf\kappa} = \kappa^{cf\kappa}$.

Corollary

$\kappa^\lambda = 2^\lambda$ or $\kappa^\lambda = \kappa$

or $\kappa^\lambda = \exists \mu$ for a μ with $cf\mu \leq \lambda < \kappa$.

- If $\kappa^\lambda > 2^\lambda . \kappa$ and $\mu = \min\{\nu : \nu^\lambda = \kappa^\lambda\}$

then $\mu^\lambda = \mu^{cf\mu}$ (see above)

κ IS A STRONG LIMIT CARDINAL IF

$2^\lambda < \kappa$ FOR $\lambda < \kappa$.

EXAMPLE: $\aleph_0$, $\beth_\omega$, ----

NOTE κ STRONG LIMIT AND $\lambda, \mu < \kappa$ THEN $\mu^\lambda < \kappa$.

AND THEN $2^\kappa = \kappa^{cf\kappa}$

INACCESSIBLE CARDINALS:

(STRONGLY) INACCESSIBLE: ~~LIM~~

UNCOUNTABLE, REGULAR, STRONG LIMIT

(WEAKLY) INACCESSIBLE:

UNCOUNTABLE, REGULAR, LIMIT

SINGULAR CARDINAL HYPOTHESIS

IF κ IS SINGULAR AND $2^{cf\kappa} < \kappa$

THEN $\kappa^{cf\kappa} = \kappa^+$

(FOLLOWS FROM GCH)

MAKES CARDINAL EXPONENTIATION RELATIVELY EASY.

- κ SINGULAR

  $2^\kappa = 2^{<\kappa}$ IF 2^μ IS EVENTUALLY CONSTANT BELOW κ

  $2^\kappa = (2^{<\kappa})^+$ OTHERWISE

- IF $\kappa \leq 2^\lambda$ THEN $\kappa^\lambda = 2^\lambda$

  IF $2^\lambda < \kappa$ AND $\lambda < cf\kappa$ THEN $\kappa^\lambda = \kappa$

  IF $2^\lambda < \kappa$ AND $cf\kappa \leq \lambda$ THEN $\kappa^\lambda = \kappa^+$

Axiom of Regularity

$$(\forall S)(S \neq \emptyset \rightarrow (\exists x \in S)(x \cap S = \emptyset))$$

T is transitive if $x \in T \rightarrow x \subseteq T$

Transitive closure

Let S be a set

Recursively: $S_0 = S$

$S_{n+1} = \bigcup S_n$

Put $T = \bigcup_{n \in \omega} S_n$.

Then T is transitive and $S \subseteq T$

Note if $S \subseteq X$ and X is transitive

then $S_n \subseteq X$ for all n

and so $T \subseteq X$

Or $T = \bigcap \{X : S \subseteq X \text{ and } X \text{ is transitive}\}$

the transitive closure of S

6.2 Every class has an $\in$-minimal element

Let $S \in C$ if $S \cap C = \emptyset$ done

otherwise let $X = TC(S) \cap C$

take $x \in X$ with $x \cap X = \emptyset$

then $x \cap C = \emptyset$

if $y \in x \cap C$ then $y \in TC(S) \cap C = X \cap C$.

Cumulative Hierarchy:

$V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$

$V_\alpha = \bigcup_{\beta < \alpha} V_\beta$ (α limit)

- V_α is transitive
- $\alpha < \beta \rightarrow V_\alpha \subseteq V_\beta$
- $\alpha \subseteq V_\alpha$

• Axiom of regularity $\leftrightarrow (\forall x)(\exists \alpha)(x \in V_\alpha)$