HAPPY-ENDING PROBLEM

E. KLEIN GIVEN FIVE POINTS IN THE PLANE THERE ARE FOUR AMONG THEM THAT FORM A CONVEX QUADRANGLE

ERDÖS - SZEKERES

FOR EACH $m \in \omega$ THERE IS $N \in \omega$ SUCH THAT AMONG ANY N POINTS IN THE PLANE THERE ARE m POINTS THAT FORM A CONVEX m-GON.

(TACIT ASSUMPTION NO THREE POINTS COLLINEAR.)

PROOF USE RAMSEY'S THEOREM (FINITE VERSION): THERE IS $N \in \omega$ SUCH THAT $N \longrightarrow (m, 5)^4$

TAKE SUCH AN N AND DEFINE $F : [N]^4 \longrightarrow 2$

BY ~~$F(\{i,j\})$~~ $=$ $F(x) = \begin{cases} 0 & \text{THE POINTS IN } x \text{ FORM A CONVEX QUADRANGLE} \\ 1 & \text{THEY DO NOT} \end{cases}$

KLEIN THERE IS NO $X \in [N]^5$ WITH $F[[X]^4] = \{1\}$

SO THERE MUST BY $Y \in [N]^m$ WITH

$$F[[Y]^4] = \{0\}$$

DONE!

NOTATION $\kappa \longrightarrow (\lambda)^n_m$ MEANS

FOR EVERY $F : [\kappa]^n \longrightarrow m$

THERE IS $H \subseteq \kappa$ SUCH THAT

$|H| = \lambda$ AND $F \restriction [H]^n$ IS CONSTANT.

(H IS F-HOMOGENEOUS)

VARIATION $\kappa \longrightarrow (\lambda, \mu)^n$: $F : [\kappa]^n \longrightarrow 2$

EITHER $H \subseteq \kappa$ WITH $|H| = \lambda$ AND $F \restriction [H]^n \equiv 0$

OR $K \subseteq \kappa$ WITH $|K| = \mu$ AND $F \restriction [K]^n \equiv 1$

RAMSEY (INFINITE)

$$\aleph_0 \longrightarrow (\aleph_0)^n_k \qquad 0 < n, k < \omega$$

$k = 2$ SUFFICES: IF $k > 2$ REPEAT THE CASE $k=2$ WITH $F \restriction [X]^n \equiv k-1$

VERSUS $F \restriction [X]^n \leq k-2$

$\vdots$

$n = 1$ EASY IF $F: \omega \to 2$ THEN F IS CONSTANT ON AN INFINITE SET.

$n = 2$ LET $F: [\omega]^2 \to 2$ BE GIVEN

LET $x_0 = 0$

$$i_0 = \begin{cases} 0 & \text{IF } X_0 = \{x > x_0 : F(\{x_0, x\}) = 0\} \text{ IS INFINITE} \\ 1 & \text{OTHERWISE SO } X_0 = \{x > x_0 : F(\{x_0, x\}) = 1\} \text{ IS INFINITE} \end{cases}$$

RECURSION

$$x_{n+1} = \min X_n$$

$$i_{n+1} = \begin{cases} 0 & \text{IF } X_{n+1} = \{x \in X_n : x > x_{n+1},\ F(\{x_{n+1}, x\}) = 0\} \text{ IS INFINITE} \\ 1 & \text{IF } X_{n+1} = \{x \in X_n : x > x_{n+1},\ F(\{x_{n+1}, x\}) = 1\} \text{ IS INFINITE (EVEN CO-FINITE).} \end{cases}$$

NOW NOTE:

IF $m < n$ THEN

$x_n \in X_m$ SO $F(\{x_m, x_n\}) = i_m$

LET i BE SUCH THAT $H = \{m : i_m = i\}$ IS INFINITE

THEN $F(\{x_m, x_n\}) = i_m = i$

WHENEVER $m, n \in H$ AND $m < n$

SO $F \restriction [\{x_m : m \in H\}]^2 \equiv i$

$n \to n+1$ SAME PROOF

$X_{-1} = \omega$

$x_0 = 0$

$X_0 \subseteq \{x : x > x_0\}$ INFINITE

AND i_0 SUCH THAT

$F(\{x_0\} \cup x) = i_0$ FOR $x \in [X_0]^n$

$\cancel{X_{n+1} \subseteq}$

$x_{k+1} = \min X_k$

$X_{k+1} \subseteq \{x : x \in X_k, x > x_{k+1}\}$

AND i_{k+1} SUCH THAT

$F(\{x_{k+1}\} \cup x) = i_{k+1}$ FOR $x \in [X_{k+1}]^n$

~~pick~~ TAKE $H \subseteq \omega$ INFINITE AND $i \in 2$ SUCH THAT $i_m = i$ FOR $m \in H$

THEN $\{x_m : m \in H\}$ IS F-HOMOGENEOUS WITH COLOUR i.

FINITE RAMSEY THEOREM

ASSUME THERE IS NO $N \in \omega$ SUCH THAT

$N \longrightarrow (m, n)^2$

SO WE HAVE A SEQUENCE $\langle F_N : N \in \omega \rangle$ SUCH THAT $F_N : [N]^2 \longrightarrow 2$

AND THERE ARE NO HOMOGENEOUS SETS OF THE RIGHT KIND.

THE SEQUENCE $\langle F_N : N \in \omega \rangle$ ~~con~~ IN THE COMPACT METRIZABLE SPACE $\{0,1\}^{[\omega]^2}$ HAS A CONVERGENT SUBSEQUENCE SAY WITH LIMIT $F : [\omega]^2 \longrightarrow 2$.

④

LET $H \subseteq \omega$ BE INFINITE AND HOMOGENEOUS FOR F

TAKE $M \in \omega$ SUCH THAT $|M \cap H| > m+n$

AND THEN $N \geq M$ SUCH THAT

$$F \restriction [M]^2 = F_N \restriction [M]^2 \quad \text{(CONVERGENCE!)}$$

BUT NOW $M \cap H$ IS HOMOGENEOUS FOR F_N

AND OF CARDINALITY AT LEAST $m+n$.

CONTRADICTION.

LIMITATIONS

$$2^{\aleph_0} \not\longrightarrow (\aleph_1)^2_2$$

LET $\triangleleft$ BE A WELL-ORDER OF $\mathbb{R}$ AND $<$ THE NORMAL ORDER

DEFINE $F(\{x,y\}) =$ "THE TRUTH VALUE OF $(x \triangleleft y \leftrightarrow x < y)$"

IF $H \subseteq \mathbb{R}$ IS SUCH THAT $F \restriction [H]^2 \equiv 1$

THEN H IS WELL-ORDERED BY $<$

AND HENCE COUNTABLE

IF $H \subseteq \mathbb{R}$ IS SUCH THAT $F \restriction [H]^2 \equiv 0$

THEN $\{-x : x \in H\}$ IS WELL-ORDERED BY $<$

AND HENCE COUNTABLE

$$2^{\aleph_0} \not\longrightarrow (3)^2_{\aleph_0}$$

LIST $\mathbb{Q}$ AS $\langle q_n : n \in \omega \rangle$

DEFINE $F : [\mathbb{R}]^2 \to \omega$ BY

$F(\{x,y\}) = \min\{n : q_n \text{ IS BETWEEN } x \text{ AND } y\}$

NOW IF $x < y < z$

THEN $F(\{x,y\}) \neq F(\{y,z\})$

SO NO TRIPLE IS F-HOMOGENEOUS

ERDÖS – RADO

$$\left(2^{\aleph_0}\right)^+ \longrightarrow (\aleph_1)^2_{\aleph_0}$$

LET $\kappa = \left(2^{\aleph_0}\right)^+$.

LET $F : [\kappa]^2 \longrightarrow \omega$ FOR $a \in \kappa$ $F_a : \kappa \setminus \{a\} \to \omega$ IS GIVEN BY $F_a(x) = F(\{a, x\})$

[SLIGHT MODIFICATION OF THE PROOF IN CLASS].

FOR $C \subseteq \kappa$ COUNTABLE AND $\varphi : C \to \omega$

WE LET $u(C, \varphi) = \min\{a \in \kappa : (\forall c \in C)(c < a) \wedge F_a \upharpoonright C = \varphi\}$

IF SUCH AN a EXISTS

OTHERWISE $u(C, \varphi) = 0$.

LET $A_0 = 2^{\aleph_0}$ (INITIAL SEGMENT OF κ !)

GIVEN A_α DEFINE

$$A_{\alpha+1} = A_\alpha \cup \{u(C, \varphi) : C \in [A_\alpha]^{\aleph_0},\ \varphi : C \to \omega\}$$

THEN $|A_{\alpha+1}| \leq |A_\alpha| + |A_\alpha|^{\aleph_0} \cdot 2^{\aleph_0}$

$\leq 2^{\aleph_0}$ (IF $|A_\alpha| \leq 2^{\aleph_0}$)

α LIMIT : $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$

FOR $\alpha \leq \omega_1$ WE HAVE $|A_\alpha| \leq 2^{\aleph_0}$

AND $|A_\alpha| = 2^{\aleph_0}$ OF COURSE

BECAUSE $A_0 \subseteq A_\alpha$.

LET $A = \bigcup_{\alpha < \omega_1} A_\alpha$ $(= A_{\omega_1})$

THEN : $\left(C \in [A]^{\aleph_0} \wedge \varphi : C \to \omega\right) \to \left(u(C, \varphi) \in A\right)$

LET $a \in \kappa$ BE SUCH THAT

$(\forall x \in A)(x < a)$.

(6)

LET $x_0 = 0$

AND GIVEN $\{x_\beta : \beta < \alpha\}$

LET $x_\alpha = u(C, \varphi)$ WHERE

$C = \{x_\beta : \beta < \alpha\}$

AND $\varphi = F_a \upharpoonright C$

NOTE THAT $x_\alpha > x_\beta$ FOR $\beta < \alpha$

<u>BECAUSE</u> $a > x_\beta$ FOR $\beta < \alpha$

(THANKS TO a WE HAVE $u(C, \varphi) > c$ FOR $c \in C$)

CONSIDER $\langle x_\alpha : \alpha < \omega_1 \rangle$

NOTE IF $\beta < \alpha$ THEN

$$F(\{x_\beta, x_\alpha\}) = F(\{x_\beta, a\})$$

CHOOSE $H \subseteq \omega_1$ UNCOUNTABLE AND $n \in \omega$

SUCH THAT $F(\{x_\beta, a\}) = n$ FOR $\beta \in H$.

NOW WE HAVE FOR $\beta < \alpha$ IN H:

$$F(\{x_\beta, x_\alpha\}) = F(\{x_\beta, a\}) = n.$$

HENCE $\{x_\beta : \beta \in H\} \cup \{a\}$ IS F-HOMOGENEOUS

THIS SET HAS CARDINALITY ω_1 BUT ORDER-TYPE $\omega_1 + 1$

SEE IF YOU CAN GET ORDER-TYPE <u>$\omega_1 + 2$</u>.

HOMEWORK: 5.17 (COMPLETE ARGUMENT)
5.18
5.27
9.1
(9.4 + 9.13) ← COUNTS AS ONE.