LEMMA

LET $n \in \omega$ $(n \geq 2)$ LET $\kappa \geq \omega$ AND LET

$$F : [(2^\kappa)^+]^n \longrightarrow \kappa$$

BE GIVEN.

THERE IS AN INCREASING SEQUENCE $\langle x_\alpha : \alpha < \kappa^+ \rangle$ IN $(2^\kappa)^+$ SUCH THAT

$$F(\{x_{\alpha_1}, \ldots, x_{\alpha_{n-1}}, x_{\alpha_n}\}) = F(\{x_{\alpha_1}, \ldots, x_{\alpha_{n-1}}, x_{\alpha_{n+1}}\})$$

WHENEVER $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < \alpha_{n+1}$.

SAME PROOF AS LAST WEEK.

BUILD A SET $A \subseteq (2^\kappa)^+$ OF CARDINALITY 2^κ SUCH THAT WHENEVER $C \subseteq 2^\kappa$ HAS CARDINALITY κ AND $\varphi : [C]^{n-1} \longrightarrow \kappa$ IS SUCH THAT THERE IS a WITH $(\forall x \in C)(x < a)$ AND $\varphi = F_a \restriction [C]^{n-1}$ $\quad F_a(x) = F(x \cup \{a\})$

THEN THERE IS SUCH AN a IN $A \setminus C$, WITH $(\forall x \in C)(x < a)$.

THEN, USING $a \in (2^\kappa)^+$ WITH $(\forall x \in A)(x < a)$ BUILD $\langle x_\alpha : \alpha < \kappa^+ \rangle$ AS BEFORE:

$x_\alpha \in A$ IS SUCH THAT

$$F_{x_\alpha} \restriction [\{x_\beta : \beta < \alpha\}]^{n-1} = F_a \restriction [\{x_\beta : \beta < \alpha\}]^{n-1}.$$

WE CALL $\{x_\alpha : \alpha < \kappa^+\}$ PRE-HOMOGENEOUS

THIS SETS THINGS UP FOR AN INDUCTIVE PROOF OF THEOREM 9.6 AND EXERCISE 9.2.

(2)

ERDŐS - DUSHNIK - MILLER

IF κ INFINITE THEN $\kappa \to (\kappa, \omega)^2$.

ASSUME $[\kappa]^2 = R \cup B$ (RED BLACK)

FOR $x \in \kappa$ PUT $B_x = \{y \in \kappa : x < y$ AND $\{x, y\} \in B\}$

CASE 1:

IN EVERY $X \subseteq \kappa$ OF CARDINALITY κ THERE IS $x \in X$ WITH $|B_x \cap X| = \kappa$.

EASY: $X_0 = \kappa$ $\quad x_0 = \min\{x \in X_0 : |B_x \cap X_0| = \kappa\}$

$X_{n+1} = X_n \cap B_{x_n}$

$x_{n+1} = \min\{x \in X_{n+1} : |B_x \cap X_{n+1}| = \kappa\}$

THEN $[\{x_n : n \in \omega\}]^2 \subseteq B$

CASE 2: THERE IS $S \subseteq \kappa$ OF CARDINALITY κ SUCH THAT $|B_x \cap S| < \kappa$ FOR ALL $x \in S$.

CASE 2a: κ IS REGULAR

EASY: RECURSIVELY CHOOSE x_α IN

$S \setminus \bigcup\{B_{x_\beta} : \beta < \alpha\}$

BY REGULARITY $|\bigcup\{B_{x_\beta} \cap S : \beta < \alpha\}|$

$\leq \sum_{\beta < \alpha} |B_{x_\beta} \cap S| < \kappa$

THEN $[\{x_\alpha : \alpha < \kappa\}]^2 \subseteq R$.

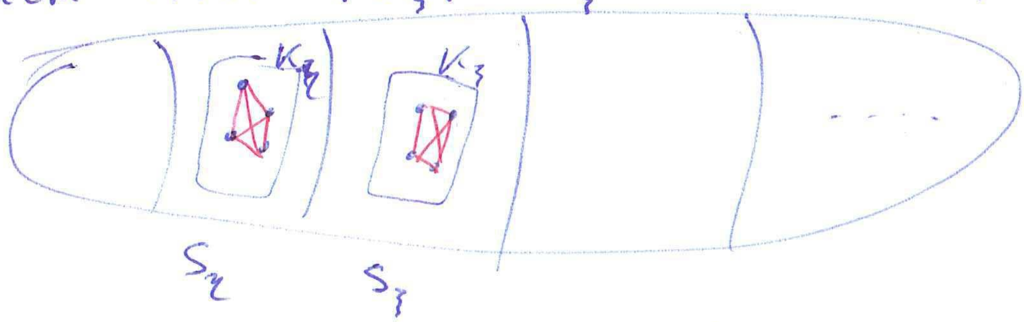
CASE 2b: κ IS SINGULAR

LET $\lambda = \operatorname{cf} \kappa$ AND LET $\langle \kappa_\xi : \xi < \lambda \rangle$ BE AN INCREASING SEQUENCE OF REGULAR CARDINALS WITH $\lambda < \kappa_0$ AND $\lim_{\xi \to \lambda} \kappa_\xi = \kappa$.

WE KNOW $\kappa_\xi \to (\kappa_\xi, \omega)^2$ FOR ALL ξ.

WE ASSUME THERE IS NO INFINITE H WITH $[H]^2 \subseteq B$.

LET $\{S_\xi : \xi < \lambda\}$ BE A PARTITION OF S SUCH THAT $|S_\xi| = \kappa_\xi$ FOR ALL ξ.



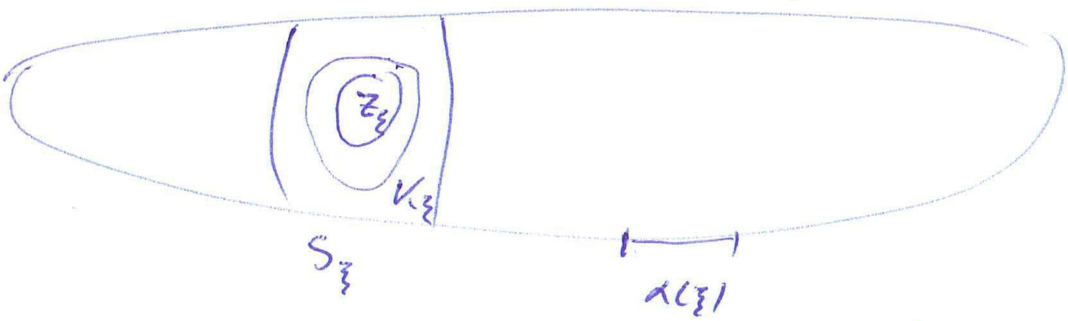
FOR EACH ξ TAKE $K_\xi \subseteq S_\xi$ OF CARDINALITY κ_ξ AND SUCH THAT $[K_\xi]^2 \subseteq R$.

FIX ξ: FOR EVERY $x \in K_\xi$ THERE IS $\alpha(x) < \lambda$ SUCH THAT $|B_x \cap S| < \kappa_{\alpha(x)}$

AS κ_ξ IS REGULAR AND $\lambda < \kappa_\xi$ THERE IS AN $\alpha(\xi) < \lambda$ SUCH THAT

$Z_\xi = \{x \in K_\xi : |B_x \cap S| < \kappa_{\alpha(\xi)}\}$

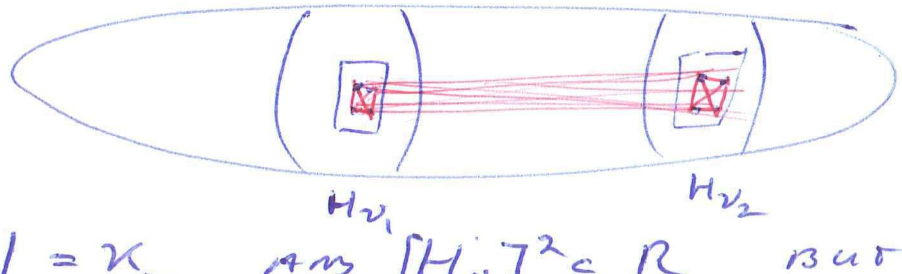
HAS CARDINALITY κ_ξ



NOTE: $|\bigcup\{|B_x \cap S| : x \in Z_\xi\}| < \kappa_{\alpha(\xi)}$ BY REGULARITY OF $\kappa_{\alpha(\xi)}$

LET $\langle \xi_\nu : \nu < \lambda \rangle$ BE AN INCREASING SEQUENCE IN λ SUCH THAT $\nu_1 < \nu_2$ IMPLIES $\alpha(\xi_{\nu_1}) < \xi_{\nu_2}$.

NOW LET $H_\nu = Z_{\xi_\nu} \setminus \bigcup\{B_x : x \in \bigcup_{\eta<\nu} Z_{\xi(\eta)}\}$



$H = \bigcup_{\nu<\lambda} H_\nu$

$|H_\nu| = \kappa_{\xi_\nu}$ AND $[H_\nu]^2 \subseteq R$ BUT ALSO IF $\nu_1 < \nu_2$, $x \in H_{\nu_1}$ AND $y \in H_{\nu_2}$ THEN $\{x, y\} \in R$

SO $[H]^2 \subseteq R$

OBVIOUS QUESTION:

IS THERE AN UNCOUNTABLE κ THAT BEHAVES LIKE ω? I.E., THAT SATISFIES $\kappa \to (\kappa)^2_2$

SUCH A κ IS CALLED <u>WEAKLY COMPACT</u>.

1. κ IS REGULAR

FOR IF $cf\,\kappa = \lambda < \kappa$ THAN PARTITION κ INTO λ SETS S_ξ $(\xi < \lambda)$ WITH $|S_\xi| < \kappa$ FOR ALL ξ.

$\{x, y\}$ IS RED IF $\{x, y\} \subseteq S_\xi$ FOR SOME ξ

$\{x, y\}$ IS BLACK OTHERWISE

NOTE $[H]^2 \subseteq$ BLACK: $|H \cap S_\xi| \leq 1$ FOR ALL ξ SO $|H| \leq \lambda$.

$[H]^2 \subseteq$ RED : $H \subseteq S_\xi$ FOR SOME ξ SO $|H| < \kappa$.

2. κ IS A STRONG LIMIT

REMEMBER $2^\lambda \not\to (\lambda^+)^2_2$ FOR ALL λ. (9.4)

SO THERE IS NO $\lambda < \kappa$ WITH $\kappa \leq 2^\lambda$.

SO, EVERY WEAKLY COMPACT CARDINAL IS INACCESSIBLE.

BACK TO CHAPTER 8

κ REGULAR UNCOUNTABLE

$C \subseteq \kappa$ IS <u>CLOSED</u> IF $\sup A \in C$ WHENEVER $A \subseteq C$ AND $\sup A$ EXISTS

$C \subseteq \kappa$ IS <u>UNBOUNDED</u> IF

$$(\forall x \in \kappa)(\exists c \in C)(x < c)$$

<u>CUB</u> IS SHORT FOR <u>CLOSED UNBOUNDED</u>

$S \subseteq \kappa$ IS STATIONARY IF $S \cap C \neq \emptyset$ FOR ALL CUB SETS

- C, D CUB $\rightarrow$ $C \cap D$ CUB
- $\{C_\alpha : \alpha < \gamma\}$ FAMILY OF CUBS $\gamma < \kappa$ THEN $\bigcap_{\alpha < \gamma} C_\alpha$ IS CUB
- $\{C_\alpha : \alpha < \kappa\}$ FAMILY OF CUBS THEN $\Delta_{\alpha < \kappa} C_\alpha$ IS CUB, WHERE $\Delta_{\alpha \in \kappa} C_\alpha = \{\delta \in \kappa : (\forall \gamma < \delta)(\delta \in C_\gamma)\}$

FODOR'S LEMMA:
IF S IS STATIONARY AND $f : S \rightarrow \kappa$ IS REGRESSIVE $[(\forall \alpha \in S)(\alpha > 0 \rightarrow f(\alpha) < \alpha)]$

THEN f IS CONSTANT ON A STATIONARY SET.

ASSUME NOT
SO $T_\gamma = \{\alpha \in S : f(\alpha) = \gamma\}$ IS NEVER STATIONARY
LET C_γ BE CUB WITH $C_\gamma \cap T_\gamma = \emptyset$.
CONSIDER $C = \Delta_{\gamma < \kappa} C_\gamma$.
C IS CUB, S IS STATIONARY, SO $S \cap C \neq \emptyset$.
SO LET $\delta \in S \cap C$.
THEN $\delta \in C_\gamma$ FOR ALL $\gamma < \delta$
SO $\delta \notin T_\gamma$ FOR ALL $\gamma < \delta$
SO $f(\delta) \geq \delta$ A CONTRADICTION.

THIS LEMMA RULES!

THE FREE SET LEMMA

LET $f : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ BE SUCH THAT $x \notin f(x)$ FOR ALL x.
THEN THERE IS $A \subseteq \mathbb{R}$ OF SIZE $2^{\aleph_0}$ SUCH THAT $y \notin f(x)$ WHENEVER $x, y \in A$.

A IS A <u>FREE</u> SET FOR f.

PROOF FOR EACH x TAKE $p_x, q_x \in \mathbb{Q}$ SUCH THAT $x \in (p_x, q_x)$ AND $(p_x, q_x) \cap f(x) = \emptyset$

SINCE $\mathrm{CF}\, 2^{\aleph_0} > \aleph_0$ THERE ARE $A \subseteq \mathbb{R}$ AND $p, q \in \mathbb{Q}$ SUCH THAT $|A| = 2^{\aleph_0}$ AND $(p,q) = (p_x, q_x)$ FOR $x \in A$.

THEN $A \subseteq (p,q)$ AND $f(x) \cap (p,q) = \emptyset$ FOR $x \in A$.

GENERAL THEOREM

LET κ BE A CARDINAL AND LET $f : \kappa^{++} \to [\kappa^{++}]^{\leq \kappa}$ BE SUCH THAT $\alpha \notin f(\alpha)$ FOR ALL α.
THERE IS (A STATIONARY) $S \subseteq \kappa^{++}$ OF SIZE κ^{++} SUCH THAT $\beta \notin f(\alpha)$ WHENEVER $\alpha, \beta \in S$.

CONSIDER THE STATIONARY SET

$$E = \{ \alpha < \kappa^{++} : \mathrm{CF}\, \alpha = \kappa^{+} \}$$

THE FUNCTION $g : E \to \kappa^{++}$ DEFINED BY

$g(\alpha) = \sup(f(\alpha) \cap \alpha \cup \{0\})$ IS REGRESSIVE

HENCE CONSTANT ON A STATIONARY SET S, WITH VALUE γ SAY.

SO $\alpha \in S \to f(\alpha) \cap \alpha \subseteq \gamma + 1$

EXERCISE 8.2: THERE IS A CUB SET C SUCH THAT FOR $\delta \in C$ WE HAVE

$$\alpha < \delta \to \sup f(\alpha) < \delta$$

CONSIDER $T = C \cap S$. T IS STATIONARY AND IF $\delta < \varepsilon$ IN T

THEN $\delta \notin f(\varepsilon)$ BECAUSE $f(\varepsilon) \cap \varepsilon \subseteq \gamma + 1 \leq \delta$.
$\varepsilon \notin f(\delta)$ BECAUSE $\sup f(\delta) < \varepsilon$.

$f : \omega_1 \to [\omega_1]^{\leq \omega}$ DEFINED BY $f(\alpha) = \alpha$ SHOWS THAT κ^{++} CAN NOT BE REPLACED BY κ^{+}
(NOT EVEN A FREE SET OF <u>SIZE 2</u>)