(6)

$E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}\,\alpha = \lambda\}$ (κ, λ REGULAR)

- E^κ_λ IS STATIONARY IN κ.

- E^κ_λ CAN BE SPLIT INTO κ MANY STATIONARY SETS

FOR $\alpha \in E^\kappa_\lambda$ LET $\langle a(\alpha,\gamma) : \gamma < \lambda \rangle$ BE INCREASING WITH $\alpha = \lim_{\gamma \to \lambda} a(\alpha,\gamma)$.

LET $S \subseteq E^\kappa_\lambda$ BE STATIONARY

(1) THERE IS AN $\eta < \lambda$ SUCH THAT FOR ALL $\beta < \kappa$ THE SET

$S_\beta = \{\alpha \in S : a(\alpha,\eta) \geq \beta\}$

IS STATIONARY.

IF NOT THEN WE HAVE β_η AND A CUB C_η SUCH THAT $a(\alpha,\eta) < \beta_\eta$ FOR $\alpha \in S \cap C_\eta$.

LET $C = \bigcap_{\gamma<\lambda} C_\gamma$. THIS IS CUB AS $\lambda < \kappa$.

LET $\beta = \sup_{\eta<\lambda} \beta_\eta$, THEN $\beta < \kappa$

BUT FOR $\alpha \in C \cap S$ WE HAVE

$$a(\alpha,\eta) < \beta \quad \text{FOR ALL } \eta.$$

(2) FIX OUR η AND DEFINE

$f(\alpha) = a(\alpha,\eta)$ FOR $\alpha \in S$.

FOR EACH β WE HAVE $T_\beta \subseteq S_\beta$ STATIONARY AND $\delta_\beta \geq \beta$ SUCH THAT

$f(\alpha) = \delta_\beta$ FOR $\alpha \in T_\beta$.

EXERCISE 8.2 THERE IS A CUB SET C SO THAT IF $\gamma \in C$ AND $\beta < \gamma$ THEN $\delta_\beta < \gamma$ AS WELL

THEN $\{T_\beta : \beta \in C\}$ IS THE FAMILY WE SEEK: IF $\beta < \gamma$ IN C

THEN $\beta \leq \delta_\beta < \gamma \leq \delta_\gamma$

SOLOVAY

IF κ IS REGULAR AND $S \subseteq \kappa$ IS STATIONARY THEN S CAN BE SPLIT INTO κ PWD STATIONARY SETS.

WE DEALT WITH THE E^{κ}_{λ} ALREADY (AND ITS SUBSETS)

LET S BE ARBITRARY

(1) $\{\alpha \in S : cf\,\alpha < \alpha\}$ IS STATIONARY
BY FODOR'S LEMMA $S \cap E^{\kappa}_{\lambda}$ IS STATIONARY
FOR SOME λ SO WE'RE DONE.

(2) THERE IS A CUB C SUCH THAT $cf\,\alpha = \alpha$
FOR $\alpha \in C \cap S$. SO WLOG $cf\,\alpha = \alpha$ FOR ALL S,
I.E., S CONSISTS OF REGULAR UNCTBLE CARDINALS

LEMMA 8.9: $T = \{\alpha \in S : S \cap \alpha$ NOT STAT. IN $\alpha\}$
IS STATIONARY.

LET C BE CUB AND C' ITS SET OF LIMIT POINTS.
C' IS ALSO CUB SO $S \cap C' \neq \emptyset$
LET $\alpha = \min S \cap C'$
THEN $C \cap \alpha$ IS CUB IN α ($\alpha \in C'$ SO $\alpha = \sup C \cap \alpha$)
AS α IS REGULAR UNCOUNTABLE $C' \cap \alpha$ IS CUB IN α TOO
BUT $(C' \cap \alpha) \cap S = \emptyset$. SO $\alpha \in T \cap C$.

WE SPLIT T.
FOR $\alpha \in T$ LET C_α BE CUB IN α WITH $C_\alpha \cap S = \emptyset$.
LET $\langle a(\alpha,\xi) : \xi < \alpha \rangle$ BE ITS ENUMERATION.

AS BEFORE: THERE IS A ξ SUCH THAT
FOR ALL $\beta < \kappa$ THE SET
$$T(\xi,\beta) = \{\alpha \in T : a(\alpha,\xi) \geq \beta\}$$
IS STATIONARY.
THEN WE CAN USE $f : \alpha \mapsto a(\alpha,\xi)$ TO SPLIT T
AS IN LEMMA 8.8.

IF NO SUCH ξ EXISTS WE HAVE FOR EACH ξ
A β_ξ AND A CUB C_ξ SUCH THAT
$$a(\alpha,\xi) < \beta_\xi$$
FOR $\alpha \in C_\xi \cap T$.
LET $C = \underset{\xi<\kappa}{\Delta} C_\xi$ AND $D = \{\gamma \in C : (\forall \xi < \gamma)(\beta_\xi < \gamma)\}$
LET $\gamma < \alpha$ IN $T \cap D$ IF $\xi < \gamma$ THEN $a(\alpha,\xi) < \beta_\xi < \gamma$
AND SO $a(\alpha,\gamma) = \gamma$ BUT $a(\alpha,\gamma) \notin T$.

TREES:

A TREE IS A PARTIALLY ORDERED SET $\langle T, < \rangle$ IN WHICH EVERY SET $\{y : y < x\}$ OF PREDECESSORS IS WELL-ORDERED.

EXAMPLES:

TAKE A SET, X, AND AN ORDINAL, α.

$$X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta$$

ORDERED BY $\subset$ (INCLUSION)

I.E., $s < t$ IF $\operatorname{dom} s \in \operatorname{dom} t$ AND $s = t \restriction \operatorname{dom} s$

IF $t \in X^\beta$ THEN $\{s : s < t\} = \{t \restriction \gamma : \gamma \in \beta\}$

NOTATION $o(x)$ = ORDER TYPE OF $\{y : y < x\}$

αTH LEVEL $T_\alpha = \{x : o(x) = \alpha\}$

HEIGHT $T = \sup\{o(x)+1 : x \in T\}$

BRANCH: MAXIMAL CHAIN (LINEARLY ORDERED SUBSET)

KÖNIG'S INFINITY LEMMA. (EXERCISE 9.5)

IF T IS A TREE OF HEIGHT ω WITH ALL LEVELS <u>FINITE</u> THEN T HAS AN INFINITE BRANCH.

SO, ..., IF T IS A TREE OF HEIGHT ω_1 WITH ALL LEVELS <u>COUNTABLE</u> DOES T HAVE A BRANCH OF LENGTH ω_1?

ANSWER (ARONSZAJN) NOT NECESSARILY.

WE CONSTRUCT AN <u>ARONSZAJN</u> TREE INSIDE $\mathbb{Q}^{<\omega_1}$: ALL ELEMENTS WILL BE INCREASING SEQUENCES.

NOTE THAT SUCH A TREE CANNOT HAVE AN ω_1-BRANCH: IT WOULD GIVE US AN INCREASING ω_1-SEQUENCE IN $\mathbb{Q}$.

RECURSIVELY WE FIND $A_\alpha \subseteq \mathbb{Q}^\alpha$ FOR $\alpha < \omega_1$ SUCH THAT

- A_α IS COUNTABLE
- IF $\beta < \alpha$, $x \in U_\beta$ AND $q > \sup x$ THEN THERE IS $y \in U_\alpha$ SUCH THAT $x \subseteq y$ AND $q \geq \sup y$.

- $A_0 = \{\emptyset\}$ THE ONLY ELEMENT OF $\mathbb{Q}^0$

- $\alpha \longrightarrow \alpha+1$:

  $A_{\alpha+1} = \{x^\frown r : x \in A_\alpha, r > \sup x\}$

  NOTE A_α COUNTABLE IMPLIES $A_{\alpha+1}$ COUNTABLE.

- α LIMIT

  LET $\langle \alpha_n : n \in \omega\rangle$ BE INCREASING AND COFINAL IN α.

  CONSIDER $\{\langle x, q\rangle : x \in \bigcup_{\beta<\alpha} A_\beta, q \in \mathbb{Q}, q > \sup x\}$

  THIS SET IS COUNTABLE.

  FIX A PAIR IN THIS SET

  LET $\langle q_n : n \in \omega\rangle$ BE AN INCREASING SEQUENCE OF RATIONALS SUCH THAT $q_0 > \sup x$ AND $\lim q_n \leq q$

  LET $m = \min\{n : o(x) < \alpha_n\}$

  USE THE ASSUMPTIONS TO CHOOSE $y_n \in A_{\alpha_n}$ FOR $n \geq m$ SUCH THAT $x \subset y_m \subset y_{m+1} \subset \cdots \subset y_n \subset y_{n+1} \cdots$

  AND $\sup y_n \leq q_n$ FOR $n \geq m$.

  THEN $y = \bigcup_{n \geq m} y_n$ IS A SEQUENCE OF LENGTH α AND $\sup y \leq \lim q_n \leq q$.

  TAKE ONE $y(x,q)$ FOR EACH PAIR $\langle x, q\rangle$

  AND LET A_α BE THE SET OF THESE $y(x,q)$.

NOW: IS THERE AN UNCOUNTABLE κ SUCH THAT KÖNIG'S INFINITY LEMMA HOLDS FOR κ?

THERE ARE MODELS WHERE $\aleph_2$ IS SUCH.

BUT, IF CH HOLDS THEN THERE ARE $\aleph_2$-ARONSZAJN TREES...

κ HAS THE TREE PROPERTY IF EVERY TREE OF HEIGHT κ AND WITH LEVELS OF CARDINALITY LESS THAN κ HAS A BRANCH OF LENGTH κ.

SO $\aleph_0$ HAS THE TREE PROPERTY.

$\aleph_1$ DOES NOT HAVE THE TREE PROPERTY

$\aleph_2$ MAY OR MAY NOT HAVE THE TREE PROPERTY...

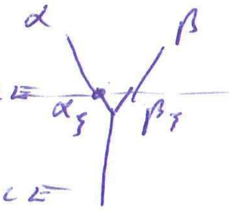
IF κ IS WEAKLY COMPACT THEN κ HAS THE TREE PROPERTY.

LET T BE A TREE, WITH ORDER $<_T$ OF HEIGHT κ AND WITH LEVELS OF CARDINALITY LESS THAN κ.

THEN $|T| = \kappa$ SO WE ASSUME $T = \kappa$ AS A SET.

EXTEND $<_T$ TO A LINEAR ORDER $\prec$

$\alpha \prec \beta$ IF $\alpha <_T \beta$



OR IF $\alpha_\xi \in \beta_\xi$ WHERE

WHEN α AND β ARE INCOMPARABLE AND ξ IS THE MINIMUM LEVEL WHERE THE PREDECESSORS OF α AND β ARE DISTINCT.

LET $F(\{\alpha,\beta\}) = \begin{cases} 1 & \text{IF } \alpha \prec \beta \leftrightarrow \alpha \in \beta \\ 0 & \text{IF } \alpha \prec \beta \leftrightarrow \beta \in \alpha \end{cases}$

LET $H \subseteq \kappa$ BE HOMOGENEOUS WITH $|H| = \kappa$.

LET $B = \{x \in \kappa : |\{\alpha \in H : x <_T \alpha\}| = \kappa\}$

NOTE EACH LEVEL INTERSECTS B

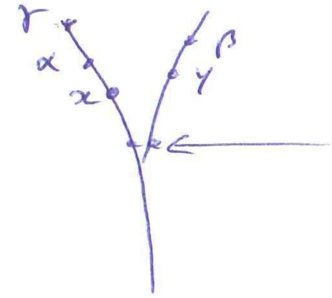
CLAIM B IS A CHAIN:

ASSUME $x, y \in B$ ARE INCOMPARABLE AND $x \prec y$.

USING THAT $x, y \in B$ TAKE $\alpha \notin \beta, \gamma \in H$

WITH $x <_T \alpha$, $y <_T \beta$, $x <_T \gamma$

AND $\alpha \in \beta \in \gamma$.



THESE DECIDE $x < y$

HENCE ALSO $\alpha < \beta$

$\gamma < \beta$

BUT THEN $F(\{\alpha, \beta\}) = 1$

AND $F(\{\beta, \gamma\}) = 0$

CONTRADICTION.

NOTE: THIS PROOF WORKS FOR ω TOO
SO RAMSEY IMPLIES KÖNIG.

IF κ IS INACCESSIBLE WITH THE TREE PROPERTY THEN κ IS WEAKLY COMPACT.

IN FACT $\kappa \to (\kappa)^2_m$ FOR $\underline{m < \kappa}$

LET $F : [\kappa]^2 \to I$ BE GIVEN WITH $|I| < \kappa$.

WE MAKE $T \subseteq I^{<\kappa}$ AS FOLLOWS:

$T = \{ t_\alpha : \alpha < \kappa \}$

- $t_0 = \emptyset$
- GIVEN $\langle t_\beta : \beta < \alpha \rangle$ WE BUILD t_α AS FOLLOWS:
  - $t_\alpha \upharpoonright 0 = \emptyset = t_0$
  - IF $t_\alpha \upharpoonright \xi$ IS KNOWN SEE IF IT IS IN $\{ t_\beta : \beta < \alpha \}$

    IF NOT STOP

    IF IT IS, SAY $t_\alpha \upharpoonright \xi = t_\beta$

    DEFINE $t_\alpha(\xi) = F(\{\beta, \alpha\})$.

AS κ IS INACCESSIBLE EACH LEVEL I^α HAS CARDINALITY LESS THAN κ.

LET B BE A BRANCH OF LENGTH κ.

AND LET $f = \bigcup B$ THEN $f : \kappa \to I$.

LET $H_i = \{ \alpha : t_\alpha \in B$, ~~$f(\alpha) = i$~~$\}$ $f(o(t_\alpha)) = i \}$

IF $\beta < \alpha$ IN H_i THEN $t_\alpha(o(t_\beta)) = i = F(\{\alpha, \beta\})$

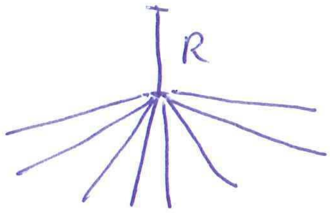
SO H_i IS HOMOGENEOUS WITH COLOUR i

ONE H_i HAS CARDINALITY κ.

AGAIN: THE PROOF WORKS FOR ω
AND SO KÖNIG IMPLIES RAMSEY.

Δ-SYSTEMS AND ALMOST DISJOINT FAMILIES.

Δ-SYSTEM : A FAMILY OF SETS, Z SUCH THAT THERE IS ONE SET R (THE ROOT) SUCH THAT $X \cap Y = R$ WHENEVER $X, Y \in Z$ AND $X \neq Y$.



EVERY <u>UNCOUNTABLE</u> FAMILY OF <u>FINITE</u> SETS CONTAINS AN UNCOUNTABLE Δ-SYSTEM.

LET Z BE THAT FAMILY.

① THERE IS $n \in \omega$ SUCH THAT

$$Z_n = \{ x \in Z : |x| = n \}$$

IS UNCOUNTABLE.

② INDUCTION ON $n \geq 1$

$n = 1$ CLEAR Z_1 IS PAIRWISE DISJOINT.

$n \to n+1$ CASE 1: THERE IS A POINT a SUCH THAT $Z' = \{ x \in Z : a \in x \}$ IS UNCOUNTABLE. APPLY THE INDUCTIVE HYPOTHESIS TO THE FAMILY $\{ x \setminus \{a\} : x \in Z' \}$

CASE 2: FOR EVERY $a \in \bigcup Z$ THE FAMILY $Z_a = \{ x \in Z : a \in x \}$ IS COUNTABLE. THEN Z HAS AN UNCOUNTABLE <u>DISJOINT</u> SUBFAMILY.

MORE GENERALLY:

IF $\kappa^{<\kappa} = \kappa$ AND W IS A FAMILY (OF SIZE κ^+) OF SETS EACH OF CARDINALITY LESS THAN κ THEN W CONTAINS A Δ-SYSTEM OF CARDINALITY κ^+.

$\mathcal{A} \subseteq \mathcal{P}(\omega)$ IS AN ALMOST DISJOINT FAMILY IF - $X \cap Y$ IS FINITE FOR ALL $X \neq Y$ IN $\mathcal{A}$

- EACH X IS INFINITE

THERE IS AN ALMOST DISJOINT FAMILY OF CARDINALITY $2^{\aleph_0}$.

MORE GENERALLY

IF κ IS REGULAR THEN $X, Y \in [\kappa]^\kappa$ ARE ALMOST DISJOINT IF $|X \cap Y| < \kappa$.

TWO FUNCTIONS f, g ~~κ~~ ON κ ARE ALMOST DISJOINT IF $|\{\alpha : f(\alpha) = g(\alpha)\}| < \kappa$.

THERE IS AN ALMOST DISJOINT FAMILY OF FUNCTIONS OF CARDINALITY κ^+

FOLLOWS FROM:

IF $\{f_\alpha : \alpha < \kappa\}$ IS AN AD FAMILY OF FUNCTIONS THEN ~~DEFINE~~ THERE IS AN f SUCH THAT $\{f\} \cup \{f_\alpha : \alpha < \kappa\}$ IS ALMOST DISJOINT:

$$f(\alpha) = \min\{\gamma \in \kappa : (\forall \beta < \alpha)(f_\beta(\alpha) \neq \gamma)\}$$

$$= \min \kappa \setminus \{f_\beta(\alpha) : \beta < \alpha\}.$$

8.2 8.5

HOMEWORK 9.3 9.5 9.10 ~~9.12~~

APPLICATION OF THE Δ-SYSTEM LEMMA

LET κ BE A CARDINAL AND
LET $\{p_\alpha : \alpha < \omega_1\}$ BE A SET OF FUNCTIONS SUCH THAT

FOR ALL α
- DOM p_α IS A FINITE SUBSET OF κ
- RAN $p_\alpha \subseteq \omega$

CLAIM THERE ARE $\alpha \neq \beta$ SUCH THAT
$p_\alpha \cup p_\beta$ IS A FUNCTION.

(1) APPLY THE Δ-SYSTEM LEMMA TO $\{\text{DOM}\, p_\alpha : \alpha < \omega_1\}$ TO GET A FINITE SET F AND $A \subseteq \omega_1$ UNCOUNTABLE SUCH THAT

$$\text{DOM}\, p_\alpha \cap \text{DOM}\, p_\beta = F$$

WHENEVER $\alpha, \beta \in A$ AND $\alpha \neq \beta$.

NOTE IF $\alpha, \beta \in A$ AND $\alpha \neq \beta$ THEN
$p_\alpha \cup p_\beta$ IS A FUNCTION IFF

$$p_\alpha \restriction F = p_\beta \restriction F$$

(2) THE SET ω^F IS COUNTABLE AND A HAS CARDINALITY $\aleph_1$ SO THERE IS ONE $\varphi \in \omega^F$ SUCH THAT $B = \{\alpha \in A : p_\alpha \restriction F = \varphi\}$ IS UNCOUNTABLE

NOW EVEN $\bigcup_{\alpha \in B} p_\alpha$ IS A FUNCTION

APPLICATION OF THIS APPLICATION:
THE PRODUCT ~~ω^κ~~ TOPOLOGY ON ω^κ SATISFIES THE <u>COUNTABLE CHAIN CONDITION</u>
I.E. EVERY PAIRWISE DISJOINT FAMILY OF OPEN SETS IS COUNTABLE.
NON-EMPTY