APPLICATION OF THE Δ-SYSTEM LEMMA

LET κ BE A CARDINAL AND
LET $\{p_\alpha : \alpha < \omega_1\}$ BE A SET
OF FUNCTIONS SUCH THAT

FOR ALL α

- DOM p_α IS A FINITE SUBSET OF κ
- RAN $p_\alpha \subseteq \omega$

CLAIM THERE ARE $\alpha \neq \beta$ SUCH THAT
$p_\alpha \cup p_\beta$ IS A FUNCTION.

① APPLY THE Δ-SYSTEM LEMMA TO $\{\mathrm{DOM}\, p_\alpha : \alpha < \omega_1\}$
TO GET A FINITE SET F AND $A \subseteq \omega_1$
UNCOUNTABLE SUCH THAT

$$\mathrm{DOM}\, p_\alpha \cap \mathrm{DOM}\, p_\beta = F$$

WHENEVER $\alpha, \beta \in A$ AND $\alpha \neq \beta$.

NOTE IF $\alpha, \beta \in A$ AND $\alpha \neq \beta$ THEN
$p_\alpha \cup p_\beta$ IS A FUNCTION IFF

$$p_\alpha \restriction F = p_\beta \restriction F$$

② THE SET ω^F IS COUNTABLE AND A HAS
CARDINALITY $\aleph_1$ SO THERE IS ONE $\varphi \in \omega^F$
SUCH THAT $B = \{\alpha \in A : p_\alpha \restriction F = \varphi\}$ IS UNCOUNTABLE

NOW EVEN $\bigcup_{\alpha \in B} p_\alpha$ IS A FUNCTION

APPLICATION OF THIS APPLICATION:

THE PRODUCT ~~ω^κ~~ TOPOLOGY ON ω^κ
SATISFIES THE <u>COUNTABLE CHAIN CONDITION</u>
I.E. EVERY PAIRWISE DISJOINT FAMILY
OF OPEN SETS IS COUNTABLE.
NON-EMPTY

ALMOST DISJOINT FAMILIES

IF κ IS A CARDINAL AND $\mathcal{A} \subseteq [\kappa]^\kappa$ THEN WE SAY $\mathcal{A}$ IS ALMOST DISJOINT IF $|X \cap Y| < \kappa$ WHENEVER $X, Y \in \mathcal{A}$ AND $X \neq Y$.

SPECIAL CASE $\kappa = \omega$:
~~\$~~$X \cap Y$ FINITE FOR ALL $X, Y \in \mathcal{A}$

EASY: IF $\mathcal{A}$ IS <u>DISJOINT</u> THEN $|\mathcal{A}| \leq \kappa$.

NOW WHAT IF $\mathcal{A}$ IS <u>ALMOST DISJOINT</u>?

SIERPIŃSKI: CONSIDER $\mathbb{Q}$ AND CHOOSE FOR EACH $x \in \mathbb{R} \setminus \mathbb{Q}$ A SEQUENCE $\langle x_n : n \in \omega \rangle$ IN $\mathbb{Q}$ WITH LIMIT x.
WRITE $S_x = \{x_n : n \in \omega\}$
- IF $x \neq y$ THEN $|S_x \cap S_y| < \omega$
- $\{S_x : x \in \mathbb{R} \setminus \mathbb{Q}\}$ HAS CARDINALITY $2^{\aleph_0}$.

OTHER EXAMPLE
$X = 2^{<\omega}$ (ALL FINITE SEQUENCES OF 0'S AND 1'S)
FOR $x \in 2^\omega$ PUT $B_x = \{x \restriction n : n < \omega\}$
AGAIN $\{B_x : x \in 2^\omega\}$ HAS SIZE $2^{\aleph_0}$
- $x \neq y \rightarrow |B_x \cap B_y| < \omega$.

THIS WORKS ALSO IF $2^{<\kappa} = \kappa$

UNCOUNTABLE REGULAR κ: THERE IS AN ADF OF CARDINALITY κ^+.
RECURSIVELY DEFINE $\langle f_\nu : \nu < \kappa^+ \rangle \subseteq \kappa^\kappa$
LEMMA IF $\{f_\gamma : \gamma < \kappa\}$ IS AD THEN SO IS $\{f_\nu : \nu < \kappa\} \cup \{g\}$
$g(\alpha) = \min \kappa \setminus \{f_\beta(\alpha) : \alpha \leq \beta\}$.

CH : THERE IS AN ADF OF SIZE $2^{\aleph_1}$ ON ω_1

NOT CH : MAYBE NOT.

EVEN MORE INTERESTING:

LET T BE A TREE OF HEIGHT ω_1 WITH COUNTABLE LEVELS.

T IS <u>KUREPA</u> IF IT HAS $\aleph_2$ MANY ω_1-BRANCHES

THE SET OF THESE BRANCHES IS AN A.D.F.

THERE IS A KUREPA TREE IFF THERE IS A KUREPA FAMILY:

- $\mathcal{F} \subseteq [\omega_1]^{\aleph_1}$
- $|\mathcal{F}| \geq \aleph_2$
- $\{X \cap \alpha : X \in \mathcal{F}\}$ IS COUNTABLE WHENEVER $\alpha < \omega_1$

$\rightarrow$ THE SET OF BRANCHES IS OUR $\mathcal{F}$.

$\leftarrow$ FOR $X \in \mathcal{F}$ $\quad f_X : \alpha \mapsto X \cap \alpha$

$T_\alpha = \{f_X \restriction \alpha : X \in \mathcal{F}\}$

$T = \bigcup_{\alpha < \omega_1} T_\alpha$ IS A TREE WITH COUNTABLE LEVELS AND $\aleph_2$ MANY BRANCHES: $\{f_X : X \in \mathcal{F}\}$

DO KUREPA FAMILIES/TREES EXIST?

UNDECIDABLE!

AXIOM OF REGULARITY

$$(\forall S)(S \neq \emptyset \rightarrow (\exists x \in S)(x \cap S = \emptyset))$$

T IS TRANSITIVE IF $x \in T \rightarrow x \subseteq T$

TRANSITIVE CLOSURE:

LET S BE A SET:

$$S_0 = S \quad , \quad S_{n+1} = \bigcup S_n$$

$$T = \bigcup_{n < \omega} S_n$$

- T IS TRANSITIVE
- A TRANSITIVE $\wedge$ $S \subseteq A$ $\rightarrow$ $T \subseteq A$

T IS THE TRANSITIVE CLOSURE OF S

EVERY NONEMPTY CLASS HAS A $\in$-MINIMAL ELEMENT.

- IF $S \in C$ AND $S \cap C = \emptyset$ DONE
- IF $S \in C$ AND $S \cap C \neq \emptyset$

  LET $X = TC(S) \cap C$

  TAKE $x \in X$ WITH $x \cap X = \emptyset$

  THEN $x \cap C = \emptyset$:

  FOR $x \subseteq TC(S)$ AND $TC(S) \cap C = \emptyset$.

$$V_0 = \emptyset \quad , \quad V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta \quad (\alpha \text{ LIMIT})$$

- EACH V_α IS TRANSITIVE
- $\alpha < \beta \rightarrow V_\alpha \subset V_\beta$
- $\alpha \subseteq V_\alpha$

AXIOM OF REGULARITY $\Leftrightarrow$ $V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$

i.e. $\forall x\ \exists \alpha\ x \in V_\alpha$

$\Rightarrow$ $C = \{x : (\forall \alpha)(x \notin V_\alpha)\}$

$x \in C$ AN $\in$-MINIMAL ELEMENT

SO $(\forall y \in x)\ (\exists \alpha)\ (y \in V_\alpha)$

REPLACEMENT $\exists \gamma\ \forall y \in x\ \exists \alpha < \gamma\ y \in V_\alpha$

SO $x \subseteq V_\gamma$ AND SO $x \in V_{\gamma+1}$ ---

$\Leftarrow$ $x \in V_\alpha$: ~~$y \in x$~~ $\delta = \min\{\beta \le \alpha : x \cap V_\beta \neq \emptyset\}$

IF $y \in V_\delta \cap x$ THEN $y \cap x = \emptyset$

FOR $\delta = \gamma + 1$ SO $y \subseteq V_\gamma$

- $\mathrm{RANK}(x) = \min\{\alpha : x \in V_{\alpha+1}\}$

- $y \in x \rightarrow \mathrm{RANK}(y) < \mathrm{RANK}(x)$

- $\mathrm{RANK}\ \alpha = \alpha$

C A CLASS : $\hat{C} = \{x \in C : (\forall z \in C)(\mathrm{RANK}\, x \le \mathrm{RANK}\, z)\}$

<u>IS A SET.</u>

DEFINITION OF CARDINALITY WITHOUT AC

BUT <u>WITH</u> REGULARITY:

$|X| = \{Y : \exists f : X \leftrightarrow Y \wedge (\forall Z\ \exists g\ g : X \leftrightarrow Z \rightarrow \mathrm{RANK}\ Z \ge \mathrm{RANK}\ Y)\}$

$\in$ - INDUCTION $\quad$ Γ TRANSITIVE (A CLASS)

Φ A FORMULA

IF - $\Phi(\emptyset)$

- $(\forall x \in \Gamma)\left((\forall y \in x)\Phi(y) \rightarrow \Phi(x) \right)$

THEN $\Phi(x)$ HOLDS FOR ALL $x \in \Gamma$

$\in$ - RECURSION

T TRANSITIVE G A FUNCTION

THEN THERE IS $F: T \to V$ SUCH

THAT $F(x) = G(F \restriction x) \quad (x \in T)$

F IS UNIQUE!

PROOF AS WITH NORMAL RECURSION

APPLICATION

A A CLASS THERE IS A UNIQUE CLASS B

SUCH THAT $(\forall x)(x \in B \Leftrightarrow x \in A \wedge x \subseteq B)$

$$G(X) = \begin{cases} 1 & \Leftrightarrow x = \chi_{\cancel{z}} \text{ FOR SOME } z \in A \\ 0 & \text{OTHERWISE} \end{cases}$$

$F(x) = G(F \restriction x) \qquad F(x) = 1 \Leftrightarrow x \in A \wedge F \restriction x \equiv 1$

$B = \{x : F(x) = 1\}$

T_1, T_2 TRANSITIVE CLASSES

$\pi : T_1 \to T_2$ AN $\in$-ISOMORPHISM

THEN $T_1 = T_2$ AND $\pi = ID$.

- $\pi \emptyset = \emptyset$ - IF $\pi(z) = z$ FOR $z \in x$

THEN $x \subseteq \pi x$ CLEAR

$\pi(x) \subseteq T_2$ ~~ALSO~~ IF $t \in \pi(x)$

THEN $t = \pi(z)$ FOR SOME z

BUT $\pi z \in \pi(x)$

IMPLIES $z \in x$ SO $t = \pi(z) = z$

$\therefore t \in x$

$\therefore \pi(x) = x \quad (x \in T_1)$.