# MODELS OF SET THEORY

IN GENERAL A SET (OR CLASS) $M$ WITH A BINARY RELATION $E$ CAN SERVE AS A STRUCTURE FOR THE LANGUAGE OF SET THEORY

WE <u>INTERPRET</u> FORMULAS AS FOLLOWS

$$(x \in y)^{M,E} \quad \text{is} \quad x \, E \, y$$

$$(x = y)^{M,E} \quad \text{is} \quad x = y$$

$$(\neg \varphi)^{M,E} \quad \text{is} \quad \neg \varphi^{M,E}$$

$$(\varphi \wedge \psi)^{M,E} \quad \text{is} \quad \varphi^{M,E} \wedge \psi^{M,E}$$

$$((\exists x)(\varphi))^{M,E} \quad \text{is} \quad (\exists x \in M)(\varphi)^{M,E}$$

(THIS SUFFICES: $\rightarrow$, $\vee$, $\leftrightarrow$ AND $\forall$ CAN BE EXPRESSED IN TERMS OF $\neg$, $\wedge$ AND $\exists$)

USE: TO SHOW THAT CERTAIN FORMULAS ARE NOT FORMALLY DERIVABLE FROM OTHERS

[LANGUAGE OF GROUPS: $*$, $\cdot^{-1}$, $e$

INTERPRET $*$ AS $+$ ; $\cdot^{-1}$ AS $-\cdot$ ; $e$ AS $0$ IN $\mathbb{Z}$

HERE $(\forall x)(\forall y)(x * y = y * x)$ IS TRUE

INTERPRET $*$ AS $\circ$ ; $\cdot^{-1}$ AS $\cdot^{-1}$ ; $e$ AS $(1)$ IN $S_3$

HERE $(\forall x)(\forall y)(x * y = y * x)$ IS FALSE

NEITHER THAT FORMULA NOR ITS NEGATION IS DERIVABLE FROM THE GROUP AXIOMS]

EXAMPLE $M = \mathbb{Z}$ ; $E$ IS $<$.

$((\forall x)(\exists y)(x \in y))^{\mathbb{Z},<}$ THEN IS

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x < y)$$

$$((\forall x)(\forall y)(\exists z)(\forall u)(u \in z \Leftrightarrow (u = x \vee u = y)))^{\mathbb{Z},<}$$

BECOMES

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\exists z \in \mathbb{Z})(\forall u \in \mathbb{Z})(u < z \Leftrightarrow (u = x \vee u = y))$$

NOT TRUE SO AXIOM OF PAIRING DOES NOT HOLD IN THIS STRUCTURE

AXIOM OF UNION:

$$(\forall x)(\exists y)(\forall z)(z \in y \Leftrightarrow (\exists u \in x)(z \in u))$$

BECOMES

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(\forall z \in \mathbb{Z})(z < y \Leftrightarrow (\exists u \in \mathbb{Z})(u < x \wedge z < u))$$

NOTE $(\exists u \in x)(\cdots)$

IS $(\exists u)(u \in x \wedge \cdots)$

THIS IS TRUE: TAKE $y = x - 1$

SO AXIOM OF UNION DOES NOT IMPLY THE AXIOM OF PAIRING.

<u>HOMEWORK</u> CHECK POWER SET AND REGULARITY IN THIS STRUCTURE.

USUALLY WE LET $E$ BE $\in$ ITSELF QUITE OFTEN, BUT NOT ALWAYS / $M$ IS TRANSITIVE.

$\varphi^{M,\in}$ IS WRITTEN $\varphi^{M}$

TO CHECK / VERIFY AXIOMS IN SUCH SITUATIONS IT HELPS IF THINGS DON'T CHANGE WHEN GOING INTO $M$.

THINGS LIKE $x=\{y,z\}$, $x=\langle y,z\rangle$, $x=\bigcup y$,
$x=\emptyset$, $x\subseteq y$, $x$ IS TRANSITIVE,
$x$ IS AN ORDINAL, $x=\omega$, ...

ARE <u>ABSOLUTE</u> FOR $M$:

WE SAY $\varphi$ IS <u>ABSOLUTE</u> FOR $M$
IF $\varphi$ AND $\varphi^M$ ARE EQUIVALENT.

$\Delta_0$-FORMULAS ARE ABSOLUTE FOR TRANSITIVE CLASSES

$\Delta_0$-FORMULAS:

- $x \in y$
- $x = y$
- IF $\varphi$ IS $\Delta_0$ THEN SO IS $\neg\varphi$.
- IF $\varphi$ AND $\psi$ ARE THEN SO IS $\varphi \wedge \psi$.
- IF $\varphi$ IS THEN SO IS $(\exists x\in y)(\varphi)$
  (AND $(\forall x\in y)(\varphi)$)

PROOF: INDUCTION ON COMPLEXITY

- $(x\in y)^M$ IS $(x\in y)$
- $(x=y)^M$ IS $(x=y$
- IF $\varphi \Leftrightarrow \varphi^M$ THEN $\neg\varphi \Leftrightarrow \neg\varphi^M$
- IF $\varphi \Leftrightarrow \varphi^M$ AND $\psi \Leftrightarrow \psi^M$ THEN
  ALSO $(\varphi\wedge\psi) \Leftrightarrow (\varphi\wedge\psi)^M$

SIMPLE LOGIC.

- ASSUME $\varphi \Leftrightarrow \varphi^M$
  CERTAINLY IF $(\exists u\in M)(u\in x \wedge \varphi^M)$
  THEN $(\exists u\in x)(\varphi^M)$
  AND $(\exists u\in x)(\varphi)$ BY IND. ASS.
  CONVERSELY IF $x\in M$ AND $(\exists u\in x)(\varphi)$
  THEN, AS $x\subseteq M$, $(\exists u\in M)(u\in x\wedge\varphi)$
  NOW USE $\varphi\Leftrightarrow\varphi^M$ TO CONCLUDE
  $(\exists u\in M)(u\in x\wedge\varphi^M)$

EXAMPLE, BY THE AXIOM OF REGULARITY,

$x$ IS AN ORDINAL $\Leftrightarrow$ $x$ IS TRANSITIVE AND LINEARLY ORDERED BY $\in$.

THE LATTER IS $\Delta_0$:

$(\forall y \in x)(\forall z \in x)(y = z \vee y \in z \vee z \in y) \wedge$

$(\forall y \in x)(\forall z \in y)(z \in x)$

$\Delta_0$-FORMULAS ARE USEFUL BECAUSE:

A TRANSITIVE $M$ SATISFIES

PAIRING IFF $\forall x \in M \, \forall y \in M \, \{x, y\} \in M$

UNION IFF $\forall x \in M \, \bigcup x \in M$

INFINITY IFF $\omega \in M$

LET $M = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$ THEN

$ZF^-$ IMPLIES $\sigma^M$ FOR ALL AXIOMS $\sigma$ OF ZF

$\uparrow$

ZF MINUS REGULARITY

SO IF $ZF^-$ IS CONSISTENT THEN SO IS ZF.

$V_\omega$ SATISFIES AL AXIOMS OF ZFC EXCEPT INFINITY

$V_{\omega+\omega}$ SATISFIES ALL AXIOMS OF ZF(C) EXCEPT REPLACEMENT:

THE WELL-ORDER $\lhd$ ON $\omega$ DEFINED BY

$n \lhd m$ IF $n$ IS EVEN AND $m$ IS ODD

OR $\begin{cases} n \equiv m \pmod 2 \\ \text{AND } n \in m \end{cases}$

~~HAS~~ IS NOT ISOMORPHIC TO ANY ORDINAL IN $V_{\omega+\omega}$.

IF $\kappa$ IS INACCESSIBLE
THEN $V_\kappa$ SATISFIES ALL OF ZF(C)

IF $\kappa$ IS THE <u>FIRST</u> INACCESSIBLE
THEN $V_\kappa \models$ "THERE ARE NO INACCESSIBLES"

SO ZF(C) DOES NOT PROVE INACCESSIBLES EXIST

ALSO WE CANNOT PROVE

THERE IS AN INACCESSIBLE

(*) IF ZFC IS CONSISTENT THEN SO IS ZFC + I

WE JUST SAW

$$ZFC + I \underset{\text{PROVES}}{\vdash} \text{"ZFC IS CONSISTENT"}$$

MODUS PONENS FROM (*) WOULD GIVE

$$ZFC + I \vdash \text{"ZFC + I IS CONSISTENT"}$$

CONTRADICTING GÖDELS SECOND INCOMPLETENESS THEOREM

FOR A REGULAR $\kappa$ WE LET

$$H(\kappa) = \{ x : |TC(x)| < \kappa \}$$

(THIS SET EXISTS : $H(\kappa) \subseteq V_\kappa$)

$$H(\omega) = V_\omega$$

$H(\omega_1)$ IS THE SET OF HEREDITARILY COUNTABLE SETS

$H(\kappa)$ SATISFIES ALL OF ZF(C) EXCEPT POSSIBLY POWER SET

CASE IN POINT:

$$H(\omega_1) \models \text{"ALL SETS ARE COUNTABLE"}$$

# ELEMENTARY SUBSTRUCTURES

THEOREM 12.1 LÖWENHEIM - SKOLEM

EVERY STRUCTURE FOR A COUNTABLE LANGUAGE HAS A COUNTABLE ELEMENTARY SUBSTRUCTURE

"IS ELEMENTARY SUBSTRUCTURE OF"

$M \prec H(\kappa)$ MEANS :

FOR EVERY FORMULA $\varphi$ AND ALL $a_1, \ldots, a_n \in M$ :

$$\varphi^M(a_1, \ldots, a_n) \leftrightarrow \varphi^{H(\kappa)}(a_1, \ldots, a_n)$$

EXTENSIONS

- IF $X \subseteq H(\kappa)$ THEN THERE IS $M \prec H(\kappa)$ WITH $|M| \leq |X| \cdot \aleph_0$
- ONE CAN ALSO ACHIEVE $M^\omega \subseteq M$ TOGETHER WITH
$$|M| \leq |X| \cdot 2^{\aleph_0}$$

LET $\kappa > \omega_1$ AND LET $M \prec H(\kappa)$ BE A COUNTABLE ELEMENTARY SUBSTRUCTURE

NOTE $M$ IS DEFINITELY NOT TRANSITIVE. (AS WE SHALL SEE MOMENTARILY.)

EASY OBSERVATIONS

LET $x \in M$ BE NONEMPTY SO

$(\exists y)(y \in x)$ IS TRUE IN $H(\kappa)$

SO IT IS TRUE IN $M$, THAT IS

$(\exists y \in M)(y \in x)$

SO IF $x \in M$ AND $x \neq \emptyset$ THEN $x \cap M \neq \emptyset$.

LIKEWISE: IF $x, y \in M$ AND $x \neq y$
THEN $x \cap M \neq y \cap M$

"$(\exists x)(\forall y \in x)(y \neq y)$" HOLDS IN $H(\kappa)$ ($x = \emptyset$)

SO $(\exists x \in M)(\forall y \in M)(y \in x \to y \neq y)$ IS TRUE

TAKE SUCH AN $x \in M$ THEN $x \cap M = \emptyset$
AND SO $x = \emptyset$.

SO $\emptyset \in M$ (NECESSARILY)

$(\exists x)(\forall y)(y \in x \leftrightarrow y = \emptyset)$ HOLDS IN $H(\kappa)$

SO $(\exists x \in M)(\forall y \in M)(y \in x \leftrightarrow y = \emptyset)$

SO $1 = \{\emptyset\} \in M$ (NECESSARILY)

INDUCTION: $(\forall n \in \omega)(n \in M)$ (EXERCISE)

ALSO $\omega \in M$ USE THE FOLLOWING FORMULA

$(\exists x)(\forall y)(y \in x \to (y = \emptyset \vee (\exists z \in x)(y = z \cup \{z\}))$

(SEE HOMEWORK PROBLEM 1.8)

THERE IS SUCH AN $x$ IN $M$ AND IT MUST BE $\omega$ ITSELF

ALSO $\omega_1 \in M$ AS THE <u>FIRST</u> ORDINAL SUCH THAT THERE IS NO BIJECTION BETWEEN IT AND $\omega$.

BUT OF COURSE $\omega_1 \not\subseteq M$.

LEMMA IF $x \in M$ AND $x$ IS COUNTABLE THEN $x \subseteq M$.

PROOF $(\exists f)(f : \omega \xrightarrow{\text{ONTO}} x)$ IS TRUE IN $H(\kappa)$

SO $(\exists f \in M)(f : \omega \xrightarrow{\text{ONTO}} x)$ TAKE SUCH AN $f$.

THEN $(\forall n \in \omega)(\exists y)(\langle n, y \rangle \in f)$ (IN $H(\kappa)$, HENCE IN $M$)

SO $(\forall n \in \omega)(\exists y \in M)(\langle n, y \rangle \in f)$

BUT $y$ IS UNIQUE SO $f(n) \in M$.

WE SEE $\{f(n) : n \in \omega\} \subseteq M$.

LET $f : \omega_1 \to \mathbb{R}$ BE CONTINUOUS

LET $\kappa \geq (2^{|\mathbb{R}|})^+$ AND LET $M \prec H(\kappa)$ BE COUNTABLE WITH $f \in M$

BY THE LEMMA $\delta_M = M \cap \omega_1$ IS AN ORDINAL.

<u>EXERCISE</u>: PROVE $\delta_M$ IS AN $\varepsilon$-NUMBER

WE SHOW $f$ IS CONSTANT ON $[\delta_M, \omega_1)$

LET $n \in \omega$ AND TAKE $\beta < \delta_M$ SUCH THAT

$$(\forall \gamma \in (\beta, \delta_M])(|f(\gamma) - f(\delta_M)| < 2^{-(n+1)})$$

THEN $(\forall \gamma, \eta \in (\beta, \delta_M))(|f(\gamma) - f(\eta)| < 2^{-n})$

BY THE TRIANGLE INEQUALITY.

SO $(\forall \gamma \in M)(\forall \eta \in M)(\gamma, \eta > \beta \to |f(\gamma) - f(\eta)| < 2^{-n})$

THAT IS $((\forall \gamma)(\forall \eta)(\gamma, \eta > \beta \to |f(\gamma) - f(\eta)| < 2^{-n}))^M$

IS TRUE, AND SO, <u>BECAUSE $\beta, f, n \in M$</u>

$$((\forall \gamma)(\forall \eta)(\gamma, \eta > \beta \to |f(\gamma) - f(\eta)| < 2^{-n}))^{H(\kappa)}$$

IS TRUE.

THIS SHOWS: IF $\gamma, \eta \geq \delta_M$ THEN

$|f(\gamma) - f(\eta)| < 2^{-n}$ FOR ALL $n$.

THAT IS $f$ IS CONSTANT ON $[\delta_M, \omega_1)$

<u>EXERCISE</u> PROVE THAT THERE IS A $\delta < \delta_M$ SUCH THAT $f$ IS CONSTANT ON $[\delta, \omega_1)$.

<u>EXERCISE</u> PROVE: IF $C$ IS CUB AND $C \in M$ THEN $\delta_M \in C$.

<u>EXERCISE</u> PROVE: IF $S \in M$ AND $\delta_M \in S$ THEN $S$ IS STATIONARY.