SILVER'S THEOREM

IF κ IS SINGULAR AND OF UNCOUNTABLE COFINALITY THEN $(\forall\alpha<\kappa)(2^\alpha=\alpha^+)$ IMPLIES $2^\kappa=\kappa^+$.

IF SCH HOLDS FOR ALL SINGULAR CARDINALS OF COUNTABLE COFINALITY THEN IT HOLDS EVERYWHERE. SCH: $2^{cf\kappa}<\kappa \to \kappa^{cf\kappa}=\kappa^+$

8.14 LET κ BE SINGULAR WITH $cf\kappa>\omega$ AND SUCH THAT $\lambda^{cf\kappa}<\kappa$ FOR ALL $\lambda<\kappa$.
LET $\langle\kappa_\alpha:\alpha<cf\kappa\rangle$ BE NORMAL WITH $\lim\kappa_\alpha=\kappa$
IF $\{\alpha<cf\kappa : \kappa_\alpha^{cf\kappa_\alpha}=\kappa_\alpha^+\}$ IS STATIONARY IN $cf\kappa$ THEN $\kappa^{cf\kappa}=\kappa^+$

IMPLIES THE SCH STATEMENT:
LET $cf\kappa>\omega$ AND LET $\langle\kappa_\alpha:\alpha<cf\kappa\rangle$ BE NORMAL WITH LIMIT κ.
IF $2^{cf\kappa}<\kappa$ THEN WLOG $\kappa_0>2^{cf\kappa}$
IF $\alpha<cf\kappa$ IS A LIMIT THEN

$$2^{cf\kappa_\alpha}\le 2^{cf\kappa}<\kappa_\alpha$$

SO IND. HYP: $\kappa_\alpha^{cf\kappa_\alpha}=\kappa_\alpha^+$ FOR ALL LIMIT $\alpha<cf\kappa$.
ALSO IF $2^{cf\kappa}<\lambda$ AND $cf\lambda<cf\kappa$
THEN $\lambda^{cf\kappa}=\lambda^+$ BY SCH AT λ
SO $\lambda^{cf\kappa}<\kappa$ FOR ALL $\lambda<\kappa$.
SO THEN ~~2^κ~~ $\kappa^{cf\kappa}=\kappa^+$ BY 8.14.

FOR EASE OF NOTATION $\kappa=\aleph_{\omega_1}$.
IF f,g HAVE DOMAIN ω_1 CALL THEM A.D.
IFF $\{\alpha : f(\alpha)=g(\alpha)\}$ IS COUNTABLE

8.15 $\aleph_\alpha^{\aleph_1}<\aleph_{\omega_1}$ FOR ALL α.
LET $F\subseteq\prod_{\alpha<\omega_1}A_\alpha$ BE ALMOST DISJOINT
ASSUME $\{\alpha : |A_\alpha|\le\aleph_{\alpha+1}\}$ IS STATIONARY
THEN $|F|\le\aleph_{\omega_1+1}$.

(2)

(1) 8.15 $\to$ 8.14

WE MUST SHOW $|\aleph_{\omega_1}^{\aleph_1}| \leq \aleph_{\omega_1+1}$ IN THIS CASE

LET $h : \omega_1 \to \aleph_{\omega_1}$ DEFINE

$f_h = \langle h_\alpha : \alpha < \omega_1 \rangle$

BY DOM $h_\alpha = \omega_1$ $\quad h_\alpha(\xi) = \begin{cases} h(\xi) & h(\xi) < \aleph_\alpha \\ 0 & \text{IF NOT} \end{cases}$

- $h \neq g \to f_h$ AND f_g ARE A.D.

SAY $h(\xi_0) \neq g(\xi_0)$

FIX α_0 SUCH THAT $h(\xi_0), g(\xi_0) < \aleph_{\alpha_0}$

FOR $\alpha \geq \alpha_0$ $\quad h_\alpha(\xi_0) = h(\xi_0) \neq g(\xi_0) = g_\alpha(\xi_0)$

SO $\quad h_\alpha \neq g_\alpha$

SO $\{f_h : h \in \aleph_{\omega_1}^{\aleph_1}\}$ IS ALMOST DISJOINT.

- $A_\alpha = \aleph_\alpha^{\aleph_1}$ IN THIS CASE

AND FOR STATIONARY MANY α WE HAVE $\aleph_\alpha^{\aleph_0} = \aleph_{\alpha+1}$ BY ASSUMPTION

SO FOR THOSE α WITH $\aleph_\alpha > 2^{\aleph_1}$

WE HAVE $\aleph_\alpha^{\aleph_1} = \aleph_{\alpha+1}$ BY 5.20

- 8.15 $|\{f_h : h \in \aleph_{\omega_1}^{\aleph_1}\}| \leq \aleph_{\omega_1+1}$

BUT $h \mapsto f_h$ IS ONE-TO-ONE

(2) PROOF OF 8.15

8.16 STEP 1 AS IN 8.15 BUT ASSUME $|A_\alpha| \leq \aleph_\alpha$ STATIONARILY OFTEN.

CONCLUSION $|F| \leq \aleph_{\omega_1}$.

SO WLOG $A_\alpha \subseteq \omega_\alpha$ STATIONARILY OFTEN

~~LET~~ f $\quad S_0 = \{\alpha < \omega_1 : \alpha \text{ IS A LIMIT} \wedge A_\alpha \subseteq \omega_\alpha\}$

LET $f \in F$: $\alpha \mapsto \min\{\beta < \alpha : f(\alpha) < \omega_\beta\}$

IS REGRESSIVE ON S_0 HENCE CONSTANT ON SOME STATIONARY $S_f \subseteq S_0$ WITH VALUE γ_f

SO ON S_f THE FUNCTION f IS BOUNDED BY $\omega_{\gamma_f} < \omega_{\omega_1}$.

WE GET A MAP $f \mapsto \langle S_f, f \restriction S_f \rangle$

- THIS MAP IS ONE-TO-ONE <u>BECAUSE</u> F IS ALMOST DISJOINT
  IF $S_f = S_g$ THEN $\{\alpha \in S_f : f(\alpha) = g(\alpha)\}$ IS CTBLE.
- THERE ARE $2^{\aleph_1}$ POSSIBILITIES FOR S
  AND, PER S, WE HAVE $\sum_{\gamma < \omega_1} \aleph_\gamma^{\aleph_1} = \sup_{\gamma < \omega_1} \aleph_\gamma^{\aleph_1} = \aleph_{\omega_1}^{\aleph_1}$
  POSSIBLE $f \restriction S$.
- $|F| \leq 2^{\aleph_1} \cdot \aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1}$

PROOF OF 8.15 ITSELF

LET $\mathcal{U}$ BE AN ULTRAFILTER THAT EXTENDS $\mathcal{C}_{\omega_1}$

SO $S \in \mathcal{U} \rightarrow S$ STATIONARY

ASSUME $A_\alpha \subseteq \omega_{\alpha+1}$ STATIONARILY OFTEN AND $\{\alpha : A_\alpha \subseteq \omega_{\alpha+1}\} \in \mathcal{U}$.

SAY $f < g$ IF $\{\alpha \in \omega_1 : f(\alpha) < g(\alpha)\} \in \mathcal{U}$

- $<$ IS TRANSITIVE
- $\{\alpha : f(\alpha) = g(\alpha)\}$ IS COUNTABLE, SO NOT IN $\mathcal{U}$
- UF: $<$ IS A LINEAR ORDER
- $F_f = \{g \in F : \{\alpha : g(\alpha) < f(\alpha)\}$ IS STATIONARY$\}$
  BY 8.16: $|F_f| \leq 2^{\aleph_1} \cdot \aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1}$
- FOR EVERY f : $|\{g \in F : g < f\}| \leq \aleph_{\omega_1}$
  AS $\{g \in F : g < f\} \subseteq F_f$.
- EXERCISE 5.3 $|F| \leq \aleph_{\omega_1 + 1}$ (HOMEWORK)

(4)

GALVIN – HAJNAL:

IF $\aleph_\alpha$ IS A SINGULAR STRONG LIMIT OF UNCOUNTABLE COFINALITY THEN $2^{\aleph_\alpha} < \aleph_\gamma$ WHERE $\gamma = (2^{|\alpha|})^+$

SPECIAL CASE: IF $\aleph_{\omega_1}$ IS A STRONG LIMIT THEN $2^{\aleph_{\omega_1}} < \aleph_\gamma$ WHERE $\gamma = (2^{\aleph_1})^+$

(IF $2^{\aleph_1} = \aleph_2$ THEN $\gamma = \aleph_3$

FOLLOWS FROM THE FOLLOWING LEMMA

24.2 ASSUME $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ WHENEVER $\alpha < \omega_1$

LET $F \subseteq \prod_{\alpha<\omega_1} A_\alpha$ BE ALMOST DISJOINT WITH $|A_\alpha| < \aleph_{\omega_1}$ FOR ALL α.

THEN $|F| < \aleph_\gamma$ WITH $\gamma = (2^{\aleph_1})^+$

DEFINE $<$ ON THE FUNCTIONS FROM ω_1 TO ω_1

BY $\varphi < \psi$ IF $\{\alpha < \omega_1 : \varphi(\alpha) \geq \psi(\alpha)\}$ IS NOT STATIONARY

NOTE $\varphi < \psi$ IF $\{\alpha < \omega_1 : \varphi(\alpha) < \psi(\alpha)\}$ CONTAINS A CUB.

CLAIM $<$ IS WELL-FOUNDED

ASSUME $\cdots < \varphi_{n+1} < \varphi_n < \cdots < \varphi_2 < \varphi_1 < \varphi_0$

WITH $C_n \subseteq \{\alpha : \varphi_{n+1}(\alpha) < \varphi_n(\alpha)\}$ CUB

LET $C = \bigcap_n C_n$ TAKE $\alpha \in C$

THEN $\cdots < \varphi_{n+1}(\alpha) < \varphi_n(\alpha) < \cdots < \varphi_1(\alpha) < \varphi_0(\alpha)$

CONTRADICTION.

WE HAVE A RANK FUNCTION $\|\cdot\|$ FOR WHICH

$$\|\varphi\| = \sup\{\|\psi\| + 1 : \psi < \varphi\}$$

E.G. $\|\varphi\| = 0$ IF $\{\alpha : \varphi(\alpha) = 0\}$ IS STATIONARY

~~24~~ 24.3 IMPLIES 24.2

24.3 ASSUMPTIONS AS BEFORE:

- $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ FOR ALL α
- $F \subseteq \prod_{\alpha < \omega_1} A_\alpha$ ALMOST DISJOINT

ALSO $\varphi : \omega_1 \to \omega_1$ IS GIVEN SUCH THAT

$$|A_\alpha| \le \aleph_{\alpha + \varphi(\alpha)}$$

FOR $\alpha < \omega_1$.

THEN $|F| \le$ ~~$\aleph_{\alpha + \|\varphi\|}$~~ $\underline{\aleph_{\omega_1 + \|\varphi\|}}$

IMPLICATION: ~~INDUCTION ON $\|\varphi\|$~~

THERE IS ~~AN~~ A φ SUCH THAT $|A_\alpha| \le \aleph_{\alpha+\varphi(\alpha)}$ FOR ALL α AND SO $|F| \le \aleph_{\omega_1 + \|\varphi\|}$.

~~BUT $|\omega_1^{\omega_1}| = 2^{\aleph_1}$ SO $\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\} \subseteq (2^{\aleph_1})^+$~~

~~BUT $|\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\}| \le 2^{\aleph_1}$~~

~~$\{\|\varphi\| : \varphi \in \omega$~~

BUT $|\omega_1^{\omega_1}| = 2^{\aleph_1}$ SO

$$|\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\}| \le 2^{\aleph_1}$$

SO $\Theta = \sup\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\} < (2^{\aleph_1})^+$

THIS SHOWS $\omega_1 + \|\varphi\| < (2^{\aleph_1})^+$ FOR ALL $\|\varphi\|$.

PROOF OF 24.3 BY INDUCTION ON $\|\varphi\|$

$\|\varphi\| = 0$ IF $\{\alpha : \varphi(\alpha) = 0\}$ IS STATIONARY AND THIS CASE IS PROVED IN 8.16 (!)

FOR THE REST OF THE PROOF WE NEED MORE: LET $S \subseteq \omega_1$ BE STATIONARY

$\varphi <_S \psi$ MEANS $\{\alpha \in S : \varphi(\alpha) \ge \psi(\alpha)\}$ IS NON-STATIONARY.

THEN $<_S$ IS WELL-FOUNDED TOO SO WE HAVE NORM $\|\cdot\|_S$ FOR EVERY S.

⑥

- $S \subseteq T \rightarrow \|\varphi\|_T \leq \|\varphi\|_S$

WHY : $\psi <_T \varphi$ IMPLIES $\psi <_S \varphi$

$(\{\alpha \in T : \varphi(\alpha) \geq \psi(\alpha)\} \supseteq \{\alpha \in S : \varphi(\alpha) \geq \psi(\alpha)\})$

SO $\{\psi : \psi <_T \varphi\} \subseteq \{\psi : \psi <_S \varphi\}$

INDUCTION HYP:

$\|\psi\|_T \leq \|\psi\|_S$ IF $\psi <_T \varphi$

$\sup\{\|\psi\|_T : \psi <_T \varphi\} \leq \sup\{\|\psi\|_S : \psi <_T \varphi\}$

$\leq \sup\{\|\psi\|_S : \psi <_S \varphi\}.$

- IN PARTICULAR $\|\varphi\| \leq \|\varphi\|_S$ ALWAYS

$(\|\varphi\| = \|\varphi\|_{\omega_1})$

- CHECK : $\|\varphi\|_{S \cup T} = \min\{\|\varphi\|_S, \|\varphi\|_T\}$

- S STAT, X NON STAT: $\|\varphi\|_S = \|\varphi\|_{S \cup X}$

- FOR $\varphi : \omega_1 \rightarrow \omega_1$ LET

$I_\varphi = \{X : X \text{ NON-STATIONARY}\}$

$\cup \{S : S \text{ STAT. AND } \|\varphi\| < \|\varphi\|_S\}$

I_φ IS AN IDEAL: BECAUSE OF THE TWO FORMULAS

- $\|\varphi\|_S = \|\varphi\|_{S \cup X}$
- $\|\varphi\|_{S \cup T} = \min\{\|\varphi\|_S, \|\varphi\|_T\}$

SO
- $A, B \in I_\varphi \rightarrow A \cup B \in I_\varphi$
- $A \in I_\varphi \wedge B \subseteq A \rightarrow B \in I_\varphi$
- PROPER: $\omega_1 \notin I_\varphi$.

- IF $\|\varphi\|$ IS A LIMIT
  THEN $S = S_\varphi = \{\alpha < \omega_1 : \varphi(\alpha) \text{ IS A SUCCESSOR}\}$ IS IN I_φ
  FOR IF $S \notin I_\varphi$ THEN $\|\varphi\| = \|\varphi\|_S$
  DEFINE ψ SUCH THAT $\varphi(\alpha) = \psi(\alpha) + 1$
  FOR $\alpha \in S$
  THEN $\|\varphi\|_S = \|\psi\|_S + 1$
  AND SO $\|\varphi\|$ WOULD BE A SUCCESSOR
  SO CERTAINLY $\{\alpha : \varphi(\alpha) \text{ IS A LIMIT}\} \notin I_\varphi$
- IF $\|\varphi\|$ IS A SUCCESSOR
  THEN $\{\alpha : \varphi(\alpha) \text{ IS A SUCCESSOR}\} \notin I_\varphi$
  BECAUSE $\{\alpha : \varphi(\alpha) \text{ IS A LIMIT}\}$ IS IN I_φ.

INDUCTIVE STEP a): $\|\varphi\|$ IS A LIMIT, $\|\varphi\| > 0$.
So $S = \{\alpha : \varphi(\alpha) > 0 \text{ AND } \varphi(\alpha) \text{ IS A LIMIT}\} \notin I_\varphi$
THIS MEANS $\|\varphi\| = \|\varphi\|_S$ SO WE WORK ON S.

ASSUME $A_\alpha \subseteq \omega_{\alpha+\varphi(\alpha)}$ FOR ALL α.
FOR $f \in F$ WE HAVE $f(\alpha) < \omega_{\alpha+\varphi(\alpha)}$ FOR ALL α
AND HENCE A ψ SUCH THAT
FOR ALL $\alpha \in S$: $f(\alpha) < \omega_{\alpha+\psi(\alpha)}$
AND $\psi(\alpha) < \varphi(\alpha)$.
FOR $\alpha \notin S$: $\psi(\alpha) = \varphi(\alpha)$
WE HAVE
- $\|\varphi\| = \|\varphi\|_S > \|\psi\|_S \geq \|\psi\|$
- $f \in F_\psi = \{g \in F : (\forall\alpha)(f(\alpha) < \omega_{\alpha+\psi(\alpha)})\}$

AND SO $F = \bigcup\{F_\psi : \|\psi\| < \|\varphi\|\}$
$|F_\psi| \leq \aleph_{\omega_1+\|\psi\|} < \aleph_{\omega_1+\|\varphi\|}$ FOR ALL ψ
SO $|F| = |\bigcup_{\|\psi\|<\|\varphi\|} F_\psi| \leq 2^{\aleph_1} \cdot \aleph_{\omega_1+\|\varphi\|} = \aleph_{\omega_1+\|\varphi\|}$.

INDUCTIVE STEP b) $\|\varphi\|$ IS A SUCCESSOR:
$S_0 = \{\alpha : \varphi(\alpha) \text{ IS A SUCCESSOR}\} \notin I_\varphi$
WRITE $\|\varphi\| = \gamma + 1$

(8)

Now, for $f \in F$, put

$$F_f = \{ g \in F : (\exists S \subseteq S_0)(S \notin I_\varphi \wedge (\forall \alpha \in S)(g(\alpha) \le f(\alpha)))\}$$

Now $F_f = \bigcup \{F_{f,S} : S \subseteq S_0,\ S \notin I_\varphi\}$

$$F_{f,S} = \{ g \in F : (\forall \alpha \in S)(g(\alpha) \le f(\alpha))\}$$

Given S define ψ_S so that

$\psi_S(\alpha) + 1 = \varphi(\alpha) \quad \alpha \in S$

$\psi_S(\alpha) = \varphi(\alpha) \quad \alpha \notin S$

Then $\|\psi\| \le \|\psi\|_S < \|\varphi\|_S = \|\varphi\| = \gamma + 1$

So $\|\psi\| \le \gamma$

$F_{f,S} \subseteq \prod_{\alpha<\omega_1} B_\alpha$ with $|B_\alpha| \le \aleph_{\alpha+\psi(\alpha)}$ for all α

so $|F_{f,S}| \le \aleph_{\omega_1+\gamma}$

And so $|F_f| \le 2^{\aleph_1} \aleph_{\omega_1+\gamma} = \aleph_{\omega_1+\gamma}$

Now build $\langle f_\xi : \xi < \theta \rangle$ recursively so that $f_\xi \notin \bigcup_{\nu<\xi} F_{f_\nu}$ for all ξ.

If $\nu < \xi$ then $\{\alpha \in S_0 : f_\xi(\alpha) \le f_\nu(\alpha)\} \in I_\varphi$

But then $\{\alpha \in S_0 : f_\nu(\alpha) < f_\xi(\alpha)\} \notin I_\varphi$

And so $\cancel{f_\nu < f_\xi}$

$$f_\nu \in F_{f_\xi}$$

Always $|F_{f_\xi}| \le \aleph_{\omega_1+\gamma}$

And $\{f_\nu : \nu < \xi\} \subseteq F_{f_\xi}$ so $\xi < \aleph_{\omega_1+\gamma+1}$

So $\theta \le \aleph_{\omega_1+\gamma+1}$

And so $F = \bigcup_{\xi<\theta} F_{f_\xi}$ and $|F| \le \aleph_{\omega_1+\gamma+1}$.

OVERVIEW OF SILVER

$8.15 \to 8.14$

CODE $\omega_{\omega_1}^{\omega_1}$ BY AN ALMOST DISJOINT FAMILY OF FUNCTIONS

$$h \longmapsto \langle h_\alpha : \alpha < \omega_1 \rangle$$

WHERE $h_\alpha(\xi) = \begin{cases} h(\xi) & \text{IF } h(\xi) < \omega_\alpha \\ 0 & \text{IF } h(\xi) \geq \omega_\alpha \end{cases}$

$8.16 \to 8.15$

LET $S = \{ \alpha > 0 : \alpha$ LIMIT ; $|A_\alpha| \leq \aleph_{\alpha+1} \}$

LET $\mathcal{U}$ BE AN ULTRAFILTER SUCH THAT S AND EVERY CUB SET BELONG TO $\mathcal{U}$.

ORDER F BY $f < g$ IF $\{ \alpha : f(\alpha) < g(\alpha) \} \in \mathcal{U}$

THEN $<$ IS A LINEAR ORDER AND

8.16 IMPLIES $|\{ g : g < f \}| \leq \aleph_{\omega_1}$ FOR ALL $f \in F$

EXERCISE 5.3 : $|F| \leq \aleph_{\omega_1+1}$

8.16 $S = \{ \alpha > 0 : \alpha$ LIMIT , $A_\alpha \leq \omega_\alpha \}$ IS STAT.

$f \in F$ INDUCES REGRESSIVE $f' : S \to \omega_1$

$$f'(\alpha) = \min \{ \beta : f(\alpha) < \omega_\beta \}$$

f' IS CONSTANT ON SOME S_f WITH VALUE β_f

BECAUSE F IS ALMOST DISJOINT THE MAP $f \longmapsto \langle S_f , f \restriction S_f \rangle$ IS INJECTIVE INTO $\mathcal{P}(\omega_1) \times \bigcup_{\beta < \omega_1} \omega_\beta^{\omega_1}$

SO $|F| \leq 2^{\aleph_1} \cdot \sup_{\beta < \omega_1} \aleph_\beta^{\aleph_1} = \aleph_{\omega_1}$

(10) OVERVIEW OF GALVIN-HAJNAL

ORDER $\omega_1^{\omega_1}$ BY $<$:

$\varphi < \psi$ IF $\{\alpha : \varphi(\alpha) < \psi(\alpha)\}$ CONTAINS A CUB

(IFF $\{\alpha : \varphi(\alpha) \geq \psi(\alpha)\} \in NS$)

- $<$ IS WELL-FOUNDED
- WE HAVE A RANK FUNCTION
  $\|\varphi\| = \sup\{\|\psi\|+1 : \psi < \varphi\}$
- NOTE $\|\varphi\| = 0 \Leftrightarrow \{\alpha : \varphi(\alpha) = 0\}$ IS STATIONARY
  ($\|\varphi\| = 0$ MEANS $\{\psi : \psi < \varphi\} = \emptyset$)

NOTATION : $\gamma = (2^{\aleph_1})^+$ THE CARDINAL SUCCESSOR OF $2^{\aleph_1}$

G-H : IF $\aleph_{\omega_1}$ IS A STRONG LIMIT
THEN $2^{\aleph_{\omega_1}} < \aleph_\gamma$

FOLLOWS FROM

24.2 : IF $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ FOR ALL $\alpha < \omega_1$
AND $F \subseteq \prod_{\alpha<\omega_1} A_\alpha$ IS ALMOST DISJOINT
WHERE $|A_\alpha| < \aleph_{\omega_1}$ FOR ALL α
THEN $|F| < \aleph_\gamma$
SAME AS "8.15 $\to$ 8.14".

24.3 IS 24.2 PARAMETRIZED:
ASSUMPTION : WE HAVE $\varphi \in \omega_1^{\omega_1}$
AND $|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$ FOR ALL α
CONCLUSION: $|F| \leq \aleph_{\omega_1+\|\varphi\|}$.

24.3 $\to$ 24.2 BECAUSE
- $\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\}$ IS AN ORDINAL
- $|\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\}| \leq 2^{\aleph_1}$

HENCE $\sup\{\|\varphi\| : \varphi \in \omega_1^{\omega_1}\} < \gamma$

SO ALWAYS $\aleph_{\omega_1+\|\varphi\|} < \aleph_\gamma$.

PROOF OF 24.3: INDUCTION ON $\|\varphi\|$

$\|\varphi\| = 0$ IS 8.16

$\|\varphi\| = 1$ IS 8.15

PROOF NEEDS EXTRA ORDERINGS

FOR EACH STATIONARY $S \subseteq \omega_1$ SAY

$\varphi <_S \psi$ IF $\{\alpha \in S : \varphi(\alpha) \geq \psi(\alpha)\} \in NS$

(IFF THERE IS A CUB C SUCH THAT $C \cap S \subseteq \{\alpha : \varphi(\alpha) < \psi(\alpha)\}$)

EACH $<_S$ IS WELL-FOUNDED AND INDUCES A RANK

$\|\varphi\|_S = \sup\{\|\psi\|_S + 1 : \psi <_S \varphi\}$

- $\|\varphi\| \leq \|\varphi\|_S$ ALWAYS BUT $<$ IS POSSIBLE
  SAY $S = \varphi^{\leftarrow}\{0\}$ AND $\varphi^{\leftarrow}\{1\} = T$ ARE BOTH STATIONARY THEN $\|\varphi\| = \|\varphi\|_S = 0$
  BUT $\|\varphi\|_T = 1$

- FOR $\varphi \in \omega_1^{\omega_1}$ LET
  $I_\varphi = NS \cup \{S : \|\varphi\| < \|\varphi\|_S\}$
  (THE IDEAL OF BAD SETS)

- LET $L_\varphi = \{\alpha : \varphi(\alpha)$ IS A LIMIT$\}$
  $S_\varphi = \{\alpha : \varphi(\alpha)$ IS A SUCCESSOR$\}$

- IF $S_\varphi \notin I_\varphi$ THEN $\|\varphi\|$ IS A SUCCESSOR
  DEFINE ψ BY $\psi(\alpha) = \begin{cases} \varphi(\alpha) & \alpha \notin S_\varphi \\ \varphi(\alpha) - 1 & \alpha \in S_\varphi \end{cases}$

  THEN $\|\varphi\| = \|\varphi\|_{S_\varphi} \overset{*}{=} \|\psi\|_{S_\varphi} + 1$ IS A SUCCESSOR

  * IF $\tau <_{S_\varphi} \varphi$ THEN ~~T~~ THERE IS A CUB C WITH $C \cap S_\varphi \subseteq \{\alpha \in S_\varphi : \tau(\alpha) \leq \psi(\alpha)\}$
    HENCE $\|\tau\|_{S_\varphi} \leq \|\psi\|_{S_\varphi}$.

- IF $L_\varphi \notin I_\varphi$ THEN $\|\varphi\|$ IS A ~~SUCCESSOR~~ LIMIT.
  WE SHOW $\|\varphi\|_{L_\varphi}$ IS A LIMIT:
  IF $\psi <_{L_\varphi} \varphi$ THEN THERE IS A CUB C SUCH THAT $L_\varphi \cap C \subseteq \{\alpha : \psi(\alpha) < \varphi(\alpha)\}$
  DEFINE τ BY $\tau(\alpha) = \begin{cases} \psi(\alpha) & \alpha \notin L_\varphi \cap C \\ \psi(\alpha)+1 & \alpha \in L_\varphi \cap C \end{cases}$

  THEN $\psi <_{L_\varphi} \tau <_{L_\varphi} \varphi$

  SO $\|\psi\|_{L_\varphi} < \|\tau\|_{L_\varphi} < \|\varphi\|_{L_\varphi}$ SO $\|\psi\|_{L_\varphi} + 1 < \|\varphi\|$.

NOW READ PAGES 7 AND 8