PROOF OF THEOREM 24.8:

LET κ BE A STRONG LIMIT OF COUNTABLE COFINALITY.

THERE IS AN INCREASING AND COFINAL SEQUENCE $\langle \lambda_n : n \in \omega \rangle$ OF REGULAR CARDINALS IN κ SUCH THAT $\langle \prod_{n<\omega} \lambda_n, <_{FIN} \rangle$ HAS TRUE COFINALITY κ^+.

SPECIAL CASE: IF $\aleph_\omega$ IS A STRONG LIMIT THEN THERE IS AN INFINITE SUBSET A OF ω SUCH THAT $\langle \prod_{n \in A} \aleph_n, <_{FIN} \rangle$ HAS TRUE COFINALITY $\aleph_{\omega+1}$.

(AS $\langle \lambda_n : n \in \omega \rangle$ IS APPARENTLY A SUBSEQUENCE OF $\langle \aleph_n : n \in \omega \rangle$).

SOME DEFINITIONS (SEE PAGES 460 AND 461 FOR GENERAL DEFINITIONS).

- $FIN = [\omega]^{<\omega}$ THE IDEAL OF FINITE SUBSETS OF ω
- $I \subseteq P(\omega)$ IS AN IDEAL IF $\{\omega \setminus X : X \in I\}$ IS A FILTER
  I.E. - $\omega \notin I$
  - $X, Y \in I \rightarrow X \cup Y \in I$
  - $X \in I, Y \subseteq X \rightarrow Y \in I$
- FOR $f, g : \omega \rightarrow On$ WE SAY
  $f <_I g$ IF $\{n : f(n) \geq g(n)\} \in I$
  $f \leq_I g$ IF $\{n : f(n) > g(n)\} \in I$
  $f =_I g$ IF $\{n : f(n) \neq g(n)\} \in I$

  WE SHALL MOSTLY USE $<_{FIN}$, $\leq_{FIN}$ AND $=_{FIN}$ BUT WE NEED GENERAL IDEALS LATER
- IF $S \subseteq On^\omega$ THEN g IS AN UPPER BOUND FOR S IF $f \leq_I g$ FOR ALL $f \in S$

  g IS A LEAST UPPER BOUND IF $g \leq_I h$ FOR EVERY UPPER BOUND h OF S.

②

IN A PARTIAL ORDER $\langle P, < \rangle$ WE SAY g IS AN <u>EXACT</u> UPPER BOUND FOR $S \subseteq P$ IF S IS A COFINAL SUBSET OF $\{f \in P : f < g\}$

COFINAL MEANS $\forall f \, \exists h \in S \; f \leq h$

COFINALITY OF P : $\min\{|S| : S \text{ IS COFINAL IN } P\}$

TRUE COFINALITY OF P :

$$\operatorname{TCF} P = \min\{|S| : S \text{ COFINAL IN } P \text{ AND } S \text{ IS A CHAIN}\}$$

NOTE TCF P NEED NOT EXIST:

FOR EXAMPLE $\langle [\omega_1]^{<\omega}, \subset \rangle$ HAS COFINALITY $\aleph_1$

BUT EACH CHAIN IS COUNTABLE

SO IT HAS NO <u>TRUE</u> COFINALITY.

OVERVIEW OF THE PROOF.

- TAKE $\langle \kappa_n : n \in \omega \rangle$ INCREASING AND COFINAL IN κ WITH EACH κ_n REGULAR.
- CONSTRUCT $\langle f_\alpha : \alpha < \kappa^+ \rangle$ IN $\prod_{n \in \omega} \kappa_n$ SUCH THAT

$$\alpha < \beta \rightarrow f_\alpha <_{FIN} f_\beta$$

- FIND AN EXACT UPPER BOUND g FOR $\langle f_\alpha : \alpha < \kappa^+ \rangle$

(NOTE: g NEED NOT BE IN $\prod_{n \in \omega} \kappa_n$)

- LET $\gamma_n = \operatorname{CF} g(n)$ FOR ALL n.

NOTE - $\gamma_n \leq g(n) \leq \kappa_n$

ALSO $\sup \gamma_n = \kappa$

LET $C_n \subseteq g(n)$ BE CLOSED AND COFINAL AND OF ORDER TYPE γ_n.

FOR $\alpha < \kappa^+$ DEFINE $\bar{f}_\alpha(n) = \min\{\gamma \in C_n : \gamma \geq f_\alpha(n)\}$

THEN $\langle \bar{f}_\alpha : \alpha < \kappa^+ \rangle$ IS $<_{FIN}$-COFINAL IN $\prod_{n \in \omega} C_n$

FINALLY LET $\langle k_n : n \in \omega \rangle$ BE INCREASING IN ω SUCH THAT $\langle \gamma_{k_n} : n \in \omega \rangle$ IS INCREASING, WITH LIMIT κ.

LET $\lambda_n = \gamma_{k_n}$ AND RESTRICT THE $\bar{f}_\alpha$ TO THE SUBPRODUCT $\prod_{n \in \omega} C_{k_n}$

(AND TRANSLATE TO $\prod_{n \in \omega} \lambda_n$ IF YOU WANT TO).

- CONSTRUCTION OF $\langle f_\alpha : \alpha < \kappa^+ \rangle$.

FOR $\alpha < \kappa$ DEFINE

$$f_\alpha(n) = \begin{cases} \alpha & \text{IF } \alpha < \kappa_n \\ 0 & \text{IF } \alpha \geq \kappa_n \end{cases}$$

NOTE IF $\alpha < \kappa$ THEN $\{n : \kappa_n \leq \alpha\}$ IS FINITE

IF $\alpha < \beta < \kappa$ THEN $\{n : f_\alpha(n) \geq f_\beta(n)\}$

$\subseteq \{n : f_\beta(n) = 0\}$

$= \{n : \beta \geq \kappa_n\}$.

SO $\langle f_\alpha : \alpha < \kappa \rangle$ IS $<_{FIN}$-INCREASING.

LET $\alpha \geq \kappa$ (BUT STILL $\alpha < \kappa^+$) AND ASSUME $\langle f_\beta : \beta < \alpha \rangle$ IS CONSTRUCTED.

VIA A BIJECTION $b : \alpha \to \kappa$ WRITE

$\alpha = \bigcup_{n \in \omega} A_n$ WHERE
- $|A_n| = \kappa_n$
- $A_n \subseteq A_{n+1}$ FOR ALL n.

DEFINE $f_\alpha(n) = \begin{cases} 0 & n = 0 \\ \sup\{f_\beta(n)+1 : \beta \in A_{n-1}\} & n > 0 \end{cases}$

SO IF $\beta \in A_n$ THEN $f_\alpha(m) \geq f_\beta(m)+1 > f_\beta(m)$ FOR ALL $m > n$.

ALSO $|A_{n-1}| = \kappa_{n-1}$ SO $f_\alpha(n) < \kappa_n$ AS κ_n IS REGULAR.

THUS $f_\alpha \in \prod_{n \in \omega} \kappa_n$ AND $f_\beta <_{FIN} f_\alpha$ FOR $\beta \in \alpha$.

- CONSTRUCTION OF g (OR RATHER: PROOF OF THE EXISTENCE OF SOME g).

START WITH $g_0 = \langle \kappa_n : n \in \omega \rangle$ AND TRY TO BUILD A $\leq_{FIN}$-DECREASING SEQUENCE $\langle g_\nu : \nu < \theta \rangle$ OF UPPER BOUNDS FOR $\langle f_\alpha : \alpha < \kappa^+ \rangle$ UNTIL IT IS NO LONGER POSSIBLE.

① THE SEQUENCE IS CERTAINLY SHORTER THAN $(2^{\aleph_0})^+$.

NOTE: IF $\alpha < \beta$ THEN $g_\beta \leq_{FIN} g_\alpha$ BUT $g_\beta \neq_{FIN} g_\alpha$ (WE WANT A NON-TRIVIAL SEQUENCE)

SO $\{n : g_\beta(n) > g_\alpha(n)\}$ IS FINITE

BUT $\{n : g_\beta(n) \neq g_\alpha(n)\}$ IS INFINITE

So for $\alpha < \beta < \theta$ the set

$\{n : g_\beta(n) < g_\alpha(n)\}$

is infinite

Define $G : [\theta]^2 \to \omega$ by

$$G(\alpha, \beta) = \min\{n : g_\beta(n) < g_\alpha(n)\}$$

If $\theta \geq (2^{\aleph_0})^+$ then there are $X \subseteq \theta$ of size $\aleph_1$ and $n \in \omega$ such that $G(\alpha,\beta) = n$ for $\{\alpha,\beta\} \in [X]^2$.

But then $\langle g_\alpha(n) : \alpha \in X \rangle$ contains (even) a decreasing ω_1-sequence of ordinals, which is impossible.

So certainly $\theta < (2^{\aleph_0})^+$.

Assume θ is a limit; we show that the sequence can be extended.

Let $A = \bigcup_{\nu < \theta} \operatorname{ran} g_\nu$ and $S = A^\omega$.

Since $|\theta| \leq 2^{\aleph_0}$ we have $|A| \leq 2^{\aleph_0}$ and $|S| \leq 2^{\aleph_0}$

Some $g \in S$ can be used to extend the sequence; we look at the g that cannot be used:

If $g \in S$ is ~~such th~~ not an upper bound for $\langle f_\alpha : \alpha < \kappa^+ \rangle$ then pick α_g such that $f_{\alpha_g} \not\leq_{\mathrm{fin}} g$

There are at most $2^{\aleph_0}$ many α_g's and $2^{\aleph_0} < \kappa < \kappa^+$ so there is an $\eta < \kappa^+$ above all α_g's.

Then: if $g \in S$ is not an upper bound for $\langle f_\alpha : \alpha < \kappa^+ \rangle$ then $f_\eta \not\leq_{\mathrm{fin}} g$.

(This one f_η takes care of all bad g.)

Define $g \in S$ by $g(n) = \min\{\gamma \in A : \gamma > f_\eta(n)\}$

~~No~~

SO WE HAVE $g \in S$ AND $f_\eta <_{FIN} g$
THEREFORE g MUST BE AN UPPER BOUND FOR $\langle f_\alpha : \alpha < \kappa^+ \rangle$.

NEXT: IF $\nu < \theta$ THEN $I = \{n : g_\nu(n) > f_\eta(n)\}$ IS COFINITE; BUT THEN FOR $n \in I$ WE HAVE, BY DEFINITION $g_\nu(n) \geq g(n)$.
AS θ IS A LIMIT $\{g_\nu(n) \neq g(n)$ INFINITELY OFTEN so $\langle g_\nu : \nu < \theta \rangle ^\frown \langle g \rangle$ IS A LONGER SEQUENCE.

CONCLUSION THE CONSTRUCTION MUST STOP AT A SUCCESSOR STEP, WITH LAST FUNCTION g.

- g IS AN EXACT UPPER BOUND:
  LET $f <_{FIN} g$ AND ASSUME $f \not<_{FIN} f_\alpha$ FOR ALL $\alpha < \kappa^+$.
  THEN EACH $A_\alpha = \{n : f(n) > f_\alpha(n)\}$ IS INFINITE.
  AGAIN USE $2^{\aleph_0} < \kappa$ TO FIND ONE $A \subseteq \omega$ SUCH THAT $\{\alpha : A_\alpha = A\}$ HAS CARDINALITY κ^+.
  THEN $f_\alpha \restriction A <_{FIN} f \restriction A$ FOR ALL $\alpha < \kappa^+$:
  GIVEN α TAKE $\beta > \alpha$ WITH $A = A_\beta$
  THEN $f_\alpha \restriction A <_{FIN} f_\beta \restriction A < f \restriction A$

  DEFINE $h(n) = \begin{cases} f(n) & n \in A \\ g(n) & n \notin A \end{cases}$

  THEN $f_\alpha <_{FIN} h$ FOR ALL α
  AND $h \leq_{FIN} g$ AND ALSO $h \neq_{FIN} g$.
  THIS CONTRADICTS THAT g WAS THE LAST FUNCTION IN THE SEQUENCE.

NOTE: THUS FAR WE ONLY USED THAT $2^{\aleph_0} < \kappa$.

⑥

- WE TURN g INTO OUR SEQUENCE $\langle \lambda_n : n \in \omega \rangle$.
  - $A = \{ n : g(n)$ IS A SUCCESSOR$\}$ IS FINITE
  FOR DEFINE $h(n) = \begin{cases} g(n)-1 & n \in A \\ g(n) & n \notin A \end{cases}$
  THEN $h \leq_I g$ AND h IS ALSO AN UPPER BOUND FOR $\langle f_\alpha : \alpha < \kappa^+ \rangle$
  HENCE $h =_I g$.
  - WLOG EVERY $g(n)$ IS A LIMIT.
  - FOR EACH n LET $C_n \subseteq g(n)$ BE CLOSED AND COFINAL ~~A~~ AND OF ORDER TYPE $cf\, g(n)$; WRITE $\gamma_n = cf\, g(n)$.
  - CLAIM $\sup_n \gamma_n = \kappa$.
  IF NOT, SAY $\sup_n \gamma_n = \mu < \kappa$, THEN
  $$\left| \prod_{n \in \omega} C_n \right| \leq \mu^{\aleph_0} < \kappa.$$
  FOR EVERY $f \in \prod_{n \in \omega} C_n$ THERE IS $\alpha_f < \kappa^+$ SUCH THAT $f <_{FIN} f_{\alpha_f}$.
  BECAUSE $\left|\prod_{n \in \omega} C_n\right| < \kappa < \kappa^+$ THERE IS ONE α SUCH THAT $f <_{FIN} f_\alpha$ FOR ALL $f \in \prod_{n \in \omega} C_n$.
  CONTRADICTION BECAUSE $\prod_{n \in \omega} C_n$ IS COFINAL IN $\prod_{n \in \omega} g(n)$.
  - LET $\langle k_n : n \in \omega \rangle$ BE AN INCREASING SEQUENCE IN ω SUCH THAT
  $\langle \gamma_{k_n} : n \in \omega \rangle$
  IS INCREASING AND COFINAL IN κ.
  LET $\lambda_n = \gamma_{k_n}$ FOR ALL n.
  - FOR $\alpha \in \kappa^+$ DEFINE $h_\alpha \in \prod_{n \in \omega} C_{k_n}$ BY
  $$h_\alpha(n) = \min\{ \gamma \in C_{k_n} : \gamma \geq f_\alpha(k_n) \}$$
  - IF $h \in \prod_{n \in \omega} C_{k_n}$ THEN $h <_{FIN} f_\alpha \restriction \{k_n : n \in \omega\}$ FOR SOME α AND SO $h <_{FIN} h_\alpha$.

So $\langle h_\alpha : \alpha < \kappa^+ \rangle$ is $<_{FIN}$-cofinal in $\prod_{n\in\omega} C_{k_n}$.

It is not necessarily $<_{FIN}$-increasing but it does have a $<_{FIN}$-increasing subsequence of length κ^+:

If $F \subseteq \prod_{n\in\omega} C_{k_n}$ has cardinality κ (or less) then there is an $\alpha < \kappa^+$ such that $h <_{FIN} f_\alpha \upharpoonright \{k_n : n\in\omega\}$ for all $h \in F$: first extend each h by zeros then bound by f_α's and use $|F| \leq \kappa$ to find one f_α.

This makes the construction possible.

Now transport $\langle h_\alpha : \alpha < \kappa^+ \rangle$ to $\prod_{n\in\omega} \lambda_n$.

Remarks about the special case $\kappa = \aleph_\omega$.

In this case $2^{\aleph_0} < \aleph_\omega$ suffices, for if $2^{\aleph_0} = \aleph_k$ then $\aleph_l^{\aleph_0} = \aleph_l$ for $l \geq k$ and this suffices for the proof of $\sup_n \gamma_n = \kappa_0$

So $2^{\aleph_0} < \aleph_\omega$ already implies the existence of a sequence $\langle k_n : n\in\omega \rangle$ such that $\prod_{n\in\omega} \aleph_{k_n}$ has ~~a cofinal $<_{FIN}$~~ a $<_{FIN}$ cofinal sequence of length $\aleph_{\omega+1}$.