THEOREM 24.18

$$\max \operatorname{pcf}\{\aleph_n\}_{n\in\omega} = 2^{\aleph_\omega}$$

PROVIDED $\aleph_\omega$ IS A STRONG LIMIT.

SINCE $\aleph_\omega$ IS A STRONG LIMIT THIS IS EQUIVALENT TO

$$\max \operatorname{pcf}\{\aleph_n\}_{n\in\omega} = \aleph_\omega^{\aleph_0}$$

NOTATION: IF A IS A SET OF REGULAR CARDINALS THEN

$$\operatorname{pcf} A = \{\operatorname{cof} D : D \text{ AN ULTRAFILTER ON } A\}$$

WHERE $\operatorname{cof} D = \operatorname{cof}(\prod A, <_D)$

(AND $f <_D g$ MEANS $\{a : f(a) < g(a)\} \in D$)

STRATEGY:

STEP 1 $\operatorname{pcf}\{\aleph_n\}_{n\in\omega}$ IS AN <u>INTERVAL</u> OF REGULAR CARDINALS

STEP 2a $|\operatorname{pcf}\{\aleph_n\}_{n\in\omega}| \leq 2^{2^{\aleph_0}}$

BECAUSE THERE ARE $2^{2^{\aleph_0}}$ ULTRAFILTERS ON ω.

STEP 2b BY ASSUMPTION $2^{2^{\aleph_0}} = \aleph_k$ FOR SOME k

OUR INTERVAL STARTS AT $\aleph_0$

SO IT MUST END <u>BEFORE</u> $\cancel{\aleph_{k+1}}$ $\aleph_{\omega_{k+1}}$

I.E., $\sup \operatorname{pcf}\{\aleph_n\}_{n\in\omega} < \cancel{\aleph_{k+1}}\ \aleph_{\omega_{k+1}}$

STEP 3 $\sup \operatorname{pcf}\{\aleph_n\}_{n\in\omega} = \aleph_\omega^{\aleph_0} = 2^{\aleph_\omega}$

SO ~~sup pcf~~

$2^{\aleph_\omega} < \aleph_{k+1}$ AND $\operatorname{cf} 2^{\aleph_\omega} > \aleph_\omega$

THEREFORE $2^{\aleph_\omega}$ IS A SUCCESSOR CARDINAL AND WE FIND <u>SUP = MAX</u>.

STEP 1

FIRST LEMMA 24.10

IF I IS AN IDEAL ON A AND $\lambda > 2^{|A|}$ REGULAR THEN EVERY $<_I$-INCREASING SEQUENCE OF LENGTH λ IN On^A HAS AN <u>EXACT</u> UPPER BOUND.

LET $\langle f_\alpha : \alpha < \lambda \rangle$ BE SUCH A SEQUENCE

LET θ BE LARGE ENOUGH

AND TAKE $M \prec H(\theta)$ WITH

- $I, F \in M$
- $|M| = 2^{|A|}$
- $M^{|A|} \subseteq M$

FOR $\alpha < \lambda$ LET $g_\alpha(a) = \min\{\beta \in M : \beta \geq f_\alpha(a)\}$

SO $g_\alpha : A \to M$ AND HENCE $g_\alpha \in M$.

AND SO, AS $|M| < \lambda$ AND λ IS REGULAR, THERE ARE $S \subseteq \lambda$ AND $f \in M$ SUCH THAT $|S| = \lambda$ AND $g_\alpha = f$ FOR $\alpha \in S$

AS $f_\alpha \leq_I f$ (EVEN $f_\alpha \leq f$) FOR $\alpha \in S$ THE FUNCTION f IS AN UPPER BOUND.

LET $h <_I f$ WE FIND α WITH $h <_I f_\alpha$.

BY ELEMENTARITY IT SUFFICES TO DEAL WITH $h \in M$:

LET $\alpha \in S$: THEN $\{a : h(a) < g_\alpha(a)\} = \{a : h(a) < f(a)\}$ ~~BELONGS TO~~

AND FOR THESE a: $h(a) < f_\alpha(a)$

BECAUSE $g_\alpha(a) = \min\{\beta \in M : \beta \geq f_\alpha(a)\}$

AND SO $h <_I f_\alpha$.

LEMMA 24.19

IF A IS AN INTERVAL OF REGULAR CARDINALS AND $\min A = (2^{|A|})^+$ THEN $\operatorname{pcf} A$ IS ALSO AN INTERVAL.

NOTE: VIA PRINCIPAL ULTRAFILTER $A \subseteq \operatorname{pcf} A$.

LET D BE AN ULTRAFILTER ON A AND LET λ BE REGULAR SUCH THAT

$\min A < \lambda < \operatorname{cof} D$

(BECAUSE $A \subseteq \operatorname{pcf} A$ WE CAN ASSUME $\lambda \geq \sup A$).

LET $\langle f_\alpha : \alpha < \operatorname{cof} D \rangle$ BE $<_D$-INCREASING AND $<_D$-COFINAL IN $\prod A$.

LET g BE AN EXACT UPPER BOUND OF THE INITIAL SEGMENT $\langle f_\alpha : \alpha < \lambda \rangle$

SO $\langle f_\alpha : \alpha < \lambda \rangle$ IS COFINAL IN $\{ \prod_a g(a), <_D \rangle$

LET $h(a) = \operatorname{cf} g(a)$ AND LET S_a BE CLOSED AND COFINAL IN $g(a)$ AND OF ORDER TYPE $h(a)$

THEN $\langle \bar{f}_\alpha : \alpha < \lambda \rangle$ IS COFINAL IN $\langle \prod_{a \in A} S_a, <_D \rangle$

WHERE $\bar{f}_\alpha(a) = \min\{ \gamma \in S_a : \gamma \geq f_\alpha(a) \}$

AND THIS TRANSLATES INTO ~~AN~~ A $<_D$-COFINAL SEQUENCE IN $\prod_{a \in A} h(a)$, SAY $\langle h_\alpha : \alpha < \lambda \rangle$.

LET $X = \{ a : h(a) \leq 2^{|A|} \}$ THEN $\left| \prod_{a \in X} h(a) \right| \leq 2^{|A|} < \lambda$

SO $X \notin D$ AND HENCE $\{ a : h(a) > 2^{|A|} \} \in D$

AS A IS AN INTERVAL WE GET $\{ a : h(a) \in A \} \in D$

SO WLOG $h(a) \in A$ FOR ALL a.

DEFINE ~~$\bar{h}_\alpha : A$~~. $\bar{h}_\alpha \in \prod A$ BY

$\bar{h}_\alpha(a) = \sup\{ h_\alpha(\beta) : h(\beta) = a \}$

NOTE $\bar{h}_\alpha \circ h \geq h_\alpha$ FOR ALL α

SO $\langle \bar{h}_\alpha \circ h : \alpha < \lambda \rangle$ IS $<_D$-COFINAL IN $\prod_{a \in A} h(a)$

AND SO $\langle \bar{h}_\alpha : \alpha < \lambda \rangle$ IS $<_E$-COFINAL IN $\prod A$,

WHERE $E = h(D) = \{ X : h^{-1}[X] \in D \}$

TURN $\langle \bar{h}_\alpha : \alpha < \lambda \rangle$ INTO A $<_E$-INCREASING AND $<_E$-COFINAL SEQUENCE.

④ STEP 2a EACH ULTRAFILTER ON ω IS A SUBSET OF $\mathcal{P}(\omega)$ HENCE AN ELEMENT OF $\mathcal{P}(\mathcal{P}(\omega))$

STEP 2b $\{\alpha : \aleph_\alpha \in pcf\{\aleph_n\}_{n\in\omega}\}$

- HAS CARDINALITY LESS THAN $\aleph_{RH}$
- CONTAINS AN INTERVAL OF SUCCESSOR ORDINALS
- SO ITS SUPREMUM IS LESS THAN $\aleph_{RH} = \omega_{RH}$.

STEP 3 PUT $\lambda = \sup pcf\{\aleph_n\}_{n\in\omega}$.

WE SHOW $\lambda = \aleph_\omega^{\aleph_0} = 2^{\aleph_\omega}$.

24.21 THERE IS $F \subseteq \prod_{n\in\omega}\aleph_n$ OF CARDINALITY λ SUCH THAT $(\forall g \in \prod_{n\in\omega}\aleph_n)(\exists f \in F)(g \leq f)$

FOR EACH ULTRAFILTER D LET $\langle f^D_\alpha : \alpha < \mathrm{cof}\, D\rangle$ BE $<_D$-COFINAL IN $\prod_{n\in\omega}\aleph_n$.

PUT $F_0 = \bigcup_D \{f^D_\alpha : \alpha < \mathrm{cof}\, D\}$

AND $F = \{f_1 \vee \dots \vee f_n : f_1, \dots, f_n \in F;\ n \in \omega\}$

$((f_1 \vee \dots \vee f_n)(a) = \max\{f_1(a), \dots, f_n(a)\})$.

LET $g \in \prod_{n\in\omega}\aleph_n$ PUT $X_f = \{n : g(n) > f(n)\}$ $(f \in F)$

IF $X_f \neq \emptyset$ FOR ALL f THEN $\{X_f : f \in F\}$ HAS THE FINITE INT. PROPERTY (BECAUSE $X_{f_1 \vee f_2} = X_{f_1} \cap X_{f_2}$).

HENCE $\{X_f : f \in F\} \subseteq D$ FOR SOME D

BUT THEN, IN PARTICULAR,

$\{n : g(n) > f^D_\alpha(n)\} \in D$

FOR ALL $\alpha < \mathrm{cof}\, D$

A CONTRADICTION.

WE HAVE $F \subseteq \prod_{n\in\omega} \aleph_n$ AND $k\in\omega$

SUCH THAT
- $2^{\aleph_0} \leq \aleph_k$
- $\lambda < \aleph_{\omega_k}$

LET θ BE LARGE ENOUGH, AND CONSIDER $\langle H(\theta), \in, \lhd \rangle$, WHERE $\lhd$ IS A WELL-ORDER.

FOR $a \in [\aleph_\omega]^{\aleph_0}$ WE BUILD A CHAIN OF ELEMENTARY SUBSTRUCTURES, ~~AS FOLL~~

$\langle M^a_\alpha : \alpha < \omega_k \rangle$, SUCH THAT
- $\omega_k \cup a \subseteq M^a_0$
- $|M^a_\alpha| = \aleph_k$
- $M^a_\alpha = \bigcup_{\beta<\alpha} M^a_\beta$ IF α IS A LIMIT
- $\alpha \to \alpha+1$: DEFINE $\chi^a_\alpha(n) = \sup M^a_\alpha \cap \omega_n$ $(n>k)$
  TAKE $f^a_\alpha \in F$ WITH
  $f^a_\alpha(n) \geq \chi^a_\alpha(n)$ $(n>k)$

  LET $M^a_{\alpha+1}$ BE SUCH THAT

$$M^a_\alpha, f^a_\alpha \in M^a_{\alpha+1}.$$

LET $M^a = \bigcup_{\alpha<\omega_k} M^a_\alpha$.

DEFINE $\chi^a(n) = \sup M^a \cap \omega_n$ $(n>k)$

$$\stackrel{!}{=} \sup_{\alpha<\omega_k} \chi^a_\alpha(n) \quad \cancel{\sup_{\alpha<\omega_k} f^a} \quad \stackrel{!}{=} \sup_{\alpha<\omega_k} f^a_\alpha(n)$$

$\chi^a_\alpha(n) \leq f^a_\alpha(n) \leq \chi^a_{\alpha+1}(n)$

CLAIM ① $\{\chi^a : a \in [\aleph_\omega]^{\aleph_0}\}$ HAS CARDINALITY AT MOST λ

CLAIM ② IF $\chi^a = \chi^b$ THEN $M^a \cap \aleph_\omega = M^b \cap \aleph_\omega$

TOGETHER: AS $[\aleph_\omega]^{\aleph_0} = \bigcup_a [M^a \cap \aleph_\omega]^{\aleph_0}$ WE HAVE

$$|\aleph_\omega^{\aleph_0}| \leq \lambda \cdot [\aleph_k]^{\aleph_0} = \lambda \cdot \aleph_k = \lambda.$$

(6)

CLAIM (1)

FIRST 24.23: THERE IS $\mathcal{S} \subseteq [\lambda]^{\aleph_1 k}$ OF CARD λ.
SUCH THAT FOR ALL $X \in [\lambda]^{\aleph_1 k}$ THERE
IS $S \in \mathcal{S}$ SUCH THAT $S \subseteq X$.

INDUCTION ON $\alpha \in [2^{\aleph_1 k}, \lambda]$ THERE IS
SUCH AN $\mathcal{S}_\alpha$ IN $[\alpha]^{\aleph_1 k}$, WITH CARD $|\alpha|$.

$\alpha = 2^{\aleph_1 k}$: $\mathcal{S}_\alpha = [2^{\aleph_1 k}]^{\aleph_1 k}$

α NOT A CARDINAL: USE BIJECTION BETWEEN
$|\alpha|$ AND α TO CREATE $\mathcal{S}_\alpha$
OUT OF $\mathcal{S}_{|\alpha|}$.

α A CARDINAL: NOTE $\alpha \leq \lambda < \aleph_{\omega k}$,
SO $\mathrm{CF}\,\alpha \neq \aleph_1 k$.
IN THIS CASE
$\mathcal{S}_\alpha = \bigcup_{\beta<\alpha} \mathcal{S}_\beta$ WORKS.

NOW ~~LET~~ FOR $a \in [\aleph_\omega]^{\aleph_0}$ ~~AND~~ LET $X_a = \{f^a_\alpha : \alpha < \omega_k\}$
WE HAVE $\mathcal{S} \subseteq [F]^{\aleph_1 k}$ SUCH THAT $|\mathcal{S}| = \lambda$
AND $(\forall X \in [F]^{\aleph_1 k})(\exists S \in \mathcal{S})(S \subseteq X)$
IN PARTICULAR: $(\exists S_a \in \mathcal{S})(S_a \subseteq X_a)$

BUT THEN $\chi^a(n) = \sup_{f \in S_a} f(n)$
THIS GIVES US AT MOST $|\mathcal{S}| = \lambda$ MANY
FUNCTIONS $\chi^a(n)$.

CLAIM 2

By induction on $n \geq k$: $M^a \cap \omega_n = M^e \cap \omega_n$.

$n = k$: $M^a \cap \omega_k = \omega_k = M^e \cap \omega_k$

$n \to n+1$ LET $\delta = \chi^a_{(n+1)} = \chi^e_{(n+1)}$

THEN $\mathrm{cf}\, \delta = \aleph_k$

THERE ARE CUB SETS $C^a \subseteq M^a \cap \delta$
AND $C^e \subseteq M^e \cap \delta$ OF ORDER TYPE ω_k.

NOTE: C^a AND C^e NEED NOT BELONG
TO M^a AND M^e RESPECTIVELY

THEN $C = C^a \cap C^e$ IS CUB IN δ AS WELL.

FOR $\gamma \in C \cap [\omega_n, \delta)$ THERE
IS A $\lhd$-FIRST BIJECTION $\pi : \omega_n \to \gamma$

THEN $\pi \in M^a$ AND $\pi \in M^e$
BY ELEMENTARITY.

So $\gamma \cap M^a = \pi[\omega_n \cap M^a]$

$= \pi[\omega_n \cap M^e] = \gamma \cap M^e$

And so

$$\delta \cap M^a = \bigcup_{\gamma \in C} \gamma \cap M^a = \bigcup_{\gamma \in C} \gamma \cap M^e = \delta \cap M^e.$$