Mastermath Exam Set Theory
17-06-2016; 14:00–17:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–16.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

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(2) 1. Given our definitions of ordered pairs ($(x,y) = \{\{x\},\{x,y\}\}$) and natural numbers ($n = \{0,\ldots,n-1\}$), which of the following is true:

A. $(0,1) = \{1,2\}$

B. $(1,2) \subseteq 3$

C. $3 \cap (0,1) = \emptyset$

D. $3 \subseteq (0,1)$

(2) 2. Consider the structure $(\mathbb{Z},<)$, for the language of set theory (so $\in$ is replaced by $<$ in every formula). Which of the following axioms of ZF **does** hold in this structure.

A. Pairing

B. Regularity

C. Union

D. Infinity

(2) 3. Assume ZF. The set V_ω, viewed as a structure for the language of Set Theory, **does not** satisfy which axiom:

A. Choice

B. Replacement

C. Power set

D. Infinity

(2) 4. Which of the following *ordinal* inequalities **does** hold:

A. $\omega^{2015} < \omega^{2016}$

B. $2^\omega < 2016^\omega$

C. $2 \cdot \omega < 2016 \cdot \omega$

D. $2 + \omega < 2016 + \omega$

(2) 5. The statement "for all sets X and Y we have $|X| \leqslant |Y|$ or $|Y| \leqslant |X|$" is

A. provable from ZF

B. provable from ZF plus the Principle of Dependent Choices

C. provable from ZFC but weaker than AC

D. equivalent to AC, provably in ZF

(2) 6. Which of the following *cardinal* inequalities **does not** hold (in ZFC):

A. $2^{\aleph_0} < 2016^{\aleph_0}$

B. $\aleph_0^{2016} < \aleph_0^{\aleph_0}$

C. $\aleph_\omega < \aleph_\omega^{\aleph_{2016}}$

D. $\aleph_{2017} \leqslant 2^{\aleph_{2016}}$

More problems on the next page.

(2) 7. Assume $2^{\aleph_n} = \aleph_{\omega+2016}$ for $n \geqslant 2016$. Then the value of $2^{\aleph_\omega}$ is

A. smaller than $\aleph_{\omega+2016}$

B. equal to $\aleph_{\omega+2016}$

C. larger than $\aleph_{\omega+2016}$

D. still undetermined

(2) 8. Which of the following statements **is not** provable in ZFC (κ, λ, and μ denote *infinite* cardinals):

A. $\aleph_{\alpha+2016}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+2016}$

B. If $\kappa \leqslant \lambda$ then $\kappa^\lambda = \aleph_0^\lambda$

C. If $\kappa < \lambda$ then $\kappa^\mu < \lambda^\mu$

D. If $\kappa \leqslant \lambda$ then $\mu^\kappa \leqslant \mu^\lambda$

(2) 9. Which of the following partition relations **is** provable in ZFC:

A. $\aleph_3 \to (\aleph_3, \aleph_3)^2$

B. $\aleph_3 \to (\aleph_3, \aleph_0)^2$

C. $\aleph_3 \to (\aleph_3, \aleph_1)^2$

D. $\aleph_3 \to (\aleph_3, \aleph_2)^2$

(2) 10. Which of the following families **is not** an ideal of sets on ω:

A. $\{A : \lim_{n\to\infty} 2^{-n}|A \cap 2^n| = 0\}$

B. $\{A : \sum_{n\in A} 2^{-n} < \infty\}$

C. $\{A : \sum_{n\in A} (n+1)^{-1} < \infty\}$

D. $\bigl\{A : (\exists k)(\forall n)\bigl(\bigl|A \cap [2^n, 2^{n+1})\bigr| \leqslant k\bigr)\bigr\}$

(2) 11. Which of the following notions **is not** expressible by means of a Δ_0-formula (assuming ZF):

A. x is an ordered pair

B. $x = \omega$

C. x is an ordinal

D. $x = \omega_1$

(2) 12. Let M a transitive model of ZFC; which of the following **is not** absolute for M:

A. x is a wellorder of y

B. $x = \omega$

C. x is the cartesian product of two sets

D. x is the cardinal number of y

13. Assume $\langle A_n : n \in \omega \rangle$ is a sequence of sets, each consisting of two points, without a choice function. (Clearly in this problem we *do not* assume the Axiom of Choice.) Define T to be the set of functions, t, such that $\operatorname{dom} t \in \omega$ and $t(i) \in A_i$ for all $i \in \operatorname{dom} t$.

(5) a. Prove that T actually is a set and indicate which axioms you use.

(4) b. Prove that T is a tree, with all levels finite.

(3) c. Define a surjective map $s : T \to \omega$.

(5) d. Prove that there is no injective map from ω to T, and in particular that there is no infinite branch in T.

More problems on the next page.

14. The Erdős-Dushnik-Miller theorem states

$$\kappa \to (\kappa, \aleph_0)^2$$

(6) a. Formulate the meaning of the statement of the theorem

(10) b. Prove the statement for the case where κ is a *regular* cardinal.

15. A *free* ultrafilter, $\mathcal{Q}$, on ω is called a Q-point if for every partition $\{P_n : n \in \omega\}$ of ω into *finite sets* there is a $U \in \mathcal{Q}$ such that $|U \cap P_n| \leqslant 1$ for all n.

(3) a. Prove: an ultrafilter on ω is Ramsey if and only if it is both a P-point and a Q-point

(4) b. Prove: $\mathcal{Q}$ is a Q-point if and only of for every strictly increasing function $f : \omega \to \omega$ there is a $U \in \mathcal{Q}$ such that $\bigl|U \cap \bigl[f(n), f(n+1)\bigr)\bigr| \leqslant 1$ for all n.

(5) c. There exist non-Q-points. *Hint*: Consider the family $\bigl\{I : I \subseteq \omega \text{ and } (\exists k)(\forall n)\bigl(\bigl|I \cap [2^n, 2^{n+1})\bigr| \leqslant k\bigr)\bigr\}$

For an infinite subset A of ω we let $c_A : \omega \to A$ denote the unique order-isomorphism.

(4) d. Prove: if $\mathcal{Q}$ is a Q-point then for every function $f : \omega \to \omega$ there is $A \in \mathcal{Q}$ such that $f(n) < c_A(n)$ for all n. *Hint*: w.l.o.g. f is strictly increasing

16. We use $H(\omega_1)$ to denote the set of *hereditarily countable sets*. That is: $x \in H(\omega_1)$ iff $\mathrm{TC}(x)$, the transitive closure of x, is countable.

(3) a. Prove: if x is a countable subset of $H(\omega_1)$ then x is an element of $H(\omega_1)$.

(3) b. Prove that $H(\omega_1)$ satisfies the Axiom (schema) of Replacement.

(3) c. Prove: for every countable transitive set x there is a well-founded relation E on ω such that x is the transitive collapse of (ω, E).

(4) d. Prove: $L_{\omega_1} \subseteq H(\omega_1) \subseteq V_{\omega_1}$.

(4) e. Prove: if $V = L$ then $L_{\omega_1} = H(\omega_1)$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END