Mastermath Exam Set Theory
30-06-2016; 10:00–13:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–16.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

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(2) 1. Given our definitions of ordered pairs ($(x, y) = \{\{x\}, \{x, y\}\}$) and natural numbers ($n = \{0, \ldots, n-1\}$), which of the following is true:

A. $(0, 0) = 1$

B. $(0, 0) \ni 1$

C. $(0, 0) \in 1$

D. none of the above

(2) 2. Consider the structure $(\mathbb{Z}, <)$, for the language of set theory (so $\in$ is replaced by $<$ in every formula). Which of the following axioms of ZF **does not** hold in this structure.

A. Extensionality

B. Power set

C. Pairing

D. Union

(2) 3. Assume ZF. The set $V_{\omega+\omega}$, viewed as a structure for the language of Set Theory, **does not** satisfy which axiom:

A. Union

B. Replacement

C. Power set

D. Infinity

(2) 4. Which of the following *ordinal* inequalities **does not** hold:

A. $\omega^{2015} < \omega^{2016}$

B. $\omega \cdot 2015 < \omega \cdot 2016$

C. $\omega + 2015 < \omega + 2016$

D. $2015^\omega < 2016^\omega$

(2) 5. Which of the following statements **is not** equivalent to the Axiom of Choice, in ZF

A. the Ultrafilter Theorem

B. Zorn's Lemma

C. the equality $|X| = |X \times X|$ holds for every infinite set X

D. the Well-ordering Theorem

(2) 6. Which of the following *cardinal* inequalities **does** hold (in ZFC):

A. $2015^{\aleph_0} < 2016^{\aleph_0}$

B. $\aleph_0^{2015} < \aleph_0^{2016}$

C. $\beth_{2015} < \beth_{2016}$

D. $\aleph_0 \cdot 2015 < \aleph_0 \cdot 2016$

More problems on the next page.

(2) 7. Assume $2^{\aleph_\alpha} = \aleph_{\omega_1+2016}$ for all countable ordinals $\alpha \geqslant 2016$. Then one possible value for $2^{\aleph_{\omega_1+1}}$ is

A. $\aleph_{\omega_1+2015}$

B. $\aleph_{\omega_1+2016}$

C. $\aleph_{\omega_1+\omega}$

D. none of the above

(2) 8. Which of the following statements **is** provable in ZFC (κ, λ, and μ denote *infinite* cardinals):

A. $\aleph_{\alpha+2016}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+2016}$

B. If $\kappa < \lambda$ then $\mu^\kappa < \mu^\lambda$

C. If $\kappa < \lambda$ then $\kappa^\mu < \lambda^\mu$

D. None of the above

(2) 9. Which of the following partition relations **is not** provable in ZFC:

A. $(2^{\aleph_{2016}})^+ \to (\aleph_{2017})^2_{\aleph_{2016}}$

B. $\aleph_{2016} \to (\aleph_{2016})^2_2$

C. $\aleph_{2016} \to (\aleph_{2016}, \aleph_0)^2$

D. $2^{\aleph_{2016}} \not\to (\aleph_0)^2_{\aleph_{2016}}$

(2) 10. Which of the following families **is** an ideal of sets on ω:

A. $\{A : \sum_{n\in A} 2^{-n} < \infty\}$

B. $\{A : \lim_{n\to\infty} 2^{-n}|A \cap 2^n| = 0\}$

C. $\{A : \sum_{n\in A}(n+1)^{-1} < 2016\}$

D. $\bigl\{A : (\forall n)\bigl(\bigl|A \cap [2^n, 2^{n+1})\bigr| \leqslant n\bigr)\bigr\}$

(2) 11. Which of the following notions **is** expressible by means of a Δ_0-formula (assuming ZF):

A. x is a well-order of y

B. x is a cardinal number

C. $x = \mathcal{P}(y)$

D. $z = x \times y$

(2) 12. Let M be a transitive model of ZFC; which of the following **is** absolute for M:

A. $x = \mathcal{P}(L_\alpha)$

B. $x = \mathbb{R}$

C. x is a cardinal number

D. x is a well-order of y

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13. In this problem we *do not* assume the Axiom of Choice. Remember that a set A is *finite* if there are $n \in \omega$ and a bijection $f : n \to A$. Define A to be *K-finite* if the following holds: every non-empty subfamily $\mathcal{S}$ of $\mathcal{P}(A)$ that satisfies "if $S \in \mathcal{S}$ and $x \in A$ then $S \cup \{x\} \in \mathcal{S}$" also satisfies $A \in \mathcal{S}$.

(6) a. Prove: every $n \in \omega$ is K-finite (hence every finite set is K-finite).

(5) b. Prove: ω is not K-finite.

(5) c. Prove: every K-finite set is finite.

More problems on the next page.

14. Ramsey's theorem states

$$\aleph_0 \to (\aleph_0)^n_k$$

(6) a. Formulate the meaning of the statement of the theorem

(4) b. Show that for fixed n it suffices to prove Ramsey's theorem for the case that $k = 2$.

Assume $k = 2$.

(7) c. Prove that the case $n = 2$ of Ramsey's theorem implies the case $n = 3$.

15. A filter, $\mathcal{F}$, on an infinite set, X, is *uniform* if $|F| = |X|$ for all $F \in \mathcal{F}$.

(4) a. Prove: every infinite set carries a uniform ultrafilter.

(4) b. Let κ be an infinite cardinal number. From the fact, proven in class, that κ carries 2^{2^κ} many ultrafilters deduce that κ also carries 2^{2^κ} many uniform ultrafilters. *Hint*: work on $\kappa \times \kappa$.

Let $\mathcal{U}$ be a free ultrafilter on ω_1.

(4) c. Prove: if $\mathcal{U}$ is not uniform then there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of $\mathcal{U}$ such that $\bigcap_n U_n = \emptyset$.

(5) d. Prove: if $\mathcal{U}$ is uniform then there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of $\mathcal{U}$ such that $\bigcap_n U_n = \emptyset$. *Hint*: consider an injection $\iota : \omega_1 \to \mathbb{R}$ and for rational q the sets $\{\alpha : \iota(\alpha) < q\}$ and $\{\alpha : \iota(\alpha) > q\}$.

16. We use $H(\omega_0)$ to denote the set of *hereditarily finite sets*. That is: $x \in H(\omega_0)$ iff $\mathrm{TC}(x)$, the transitive closure of x, is finite.

(4) a. Prove: if x is a finite subset of $H(\omega_0)$ then x is an element of $H(\omega_0)$.

(4) b. Prove that $H(\omega_0)$ satisfies the Axiom (schema) of Replacement.

(4) c. Prove: $L_\omega \subseteq H(\omega_0) \subseteq V_\omega$.

(4) d. Prove: $L_\omega = V_\omega$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END