Partially Ordered Sets
Author(s): Ben Dushnik and E. W. Miller
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PARTIALLY ORDERED SETS.*[1]

By Ben Dushnik and E. W. Miller.

1. Introduction.

1.1. By a *system* is meant a set S together with a binary relation $R(x, y)$ which may hold for certain pairs of elements x and y of S. The relation $R(x, y)$ is read "x precedes y" and is written "$x < y$." A system is called a *partial order* if the following conditions are satisfied. (1) If $x < y$, then $y \nless x$; and (2) if $x < y$ and $y < z$, then $x < z$.

A partial order defined on a set S is called a *linear order* if every two distinct elements x and y of S are comparable, i. e., if $x < y$ or $y < x$. If the partial order P and the linear order L are both defined on the same set of elements, and if every ordered pair in P occurs in L, then L will be called a *linear extension* of P.

1.2. If P and Q are two systems on the same set of elements S, then $A = P + Q = Q + P$ will denote the system which contains those and only those ordered pairs which occur in either P or Q. Likewise $P - Q$ will denote the system which contains those and only those ordered pairs which occur in P but not in Q. The system which consists of all ordered pairs which occur in both P and Q will be denoted by $P \cdot Q$. More generally, if $P_1, P_2, \cdots, P_\alpha, \cdots$ are systems on S, then ΠP_α will denote the system which consists of all ordered pairs common to all the systems P_α. It is easily seen that ΠP_α is a partial order if each system P_α is a partial order. On the other hand, it is clear that both P and Q can be partial orders without the same being true of either $P + Q$ or $P - Q$.

2. The dimension of a partial order.

2.1. Let S be any set, and let $\mathcal{K}$ be any collection of linear orders, each defined on all of S. We define a partial order P on S as follows. For any two elements x_1 and x_2 of S we put $x_1 < x_2$ (in P) if and only if $x_1 < x_2$ in every linear order of the collection $\mathcal{K}$; in other words, if $\mathcal{K} = \{L_\alpha\}$, we have $P = \Pi L_\alpha$. A partial order so obtained will be said to be *realized* by the linear orders of $\mathcal{K}$.

* Received December 8, 1940.

[1] Portions of this paper were presented to the American Mathematical Society on April 12 and November 22, 1940, under the titles "On partially ordered sets" and "On the dimension of a partial order." We are indebted to S. Eilenberg for suggestions which enabled us to simplify several proofs and improve the form of several definitions.

2. 2. By the *dimension*[2] of a partial order P defined on a set S is meant the smallest cardinal number $\mathfrak{m}$ such that P is realized by $\mathfrak{m}$ linear orders on S.

2. 3. We shall make use of the following lemma in showing that every partial order has a dimension.

LEMMA 2. 31.[3] *Every partial order P possesses a linear extension L. Moreover, if a and b are any two non-comparable elements of P, there exists an extension L_1 in which $a < b$ and an extension L_2 in which $b < a$.*

We now prove the following theorem.

THEOREM 2. 32. *If P is any partial order on a set S, then there exists a collection $\mathcal{K}$ of linear orders on S which realize P.*

Proof. If every two elements of P are comparable, then P is a linear order L and is realized by the single linear order L. If P contains non-comparable elements, then, for every non-comparable pair a and b, let $\mathcal{K}$ contain the corresponding linear extensions L_1 and L_2 mentioned in Lemma 2. 31. It is clear that P is realized by the linear orders in $\mathcal{K}$.

In light of the proof of Theorem 2. 32, the following theorem is now obvious.

THEOREM 2. 33. *Let P be any partial order on a set S. If S is finite, then the dimension of P is finite. If $\bar{\bar{S}} = \mathfrak{m}$, where $\mathfrak{m}$ is a transfinite cardinal, then the dimension of P is $\leqq \mathfrak{m}$.*

2. 4. The procedure employed in **2. 1** for defining a partial order may be formulated in the following slightly different way. Let S be any set, and let $L_1, L_2, \cdots, L_\alpha, \cdots, (\alpha < \beta)$ be a series of linear orders. (We do not require that the elements of L_α be elements of S). Let $f_1, f_2, \cdots, f_\alpha, \cdots, (\alpha < \beta)$, be a series of single-valued functions, each defined on S, each having a single-valued inverse, and such that $f_\alpha(S) \subset L_\alpha$ for every $\alpha < \beta$. We define a partial order P on S as follows. For any two elements x_1 and x_2 of S we put $x_1 < x_2$ (in P) if and only if $f_\alpha(x_1) < f_\alpha(x_2)$ for every $\alpha < \beta$. A partial order so obtained may be said to be realized by the functions f_α, and the dimension of a given partial order P may be defined as the smallest cardinal number $\mathfrak{m}$ such that P is realized by $\mathfrak{m}$ functions. It is clear that the above is nothing more than a reformulation of what appears in **2. 1** and **2. 2.**

[2] It will be noticed that the term "dimension" is here used in a different sense from that employed by Garrett Birkhoff in his book, *Lattice Theory*, American Mathematical Society Colloquium Publications.

[3] For a proof of this important result, see Edward Szpilrajn, "Sur l'extension de l'ordre partiel," *Fundamenta Mathematicae*, vol. 16 (1930), pp. 386-389.

3. Reversible partial orders.

3.1. Let P and Q be two partial orders on the same set of elements S, and suppose that every pair of distinct elements of S is ordered in just one of these partial orders; in such a case we shall say that P and Q are *conjugate* partial orders. A partial order will be called *reversible* if and only if it has a conjugate. Examples of reversible and irreversible partial orders will be given later.

3.2. A familiar example of a partial order is furnished by any family $\mathfrak{F}$ of subsets S of a given set E, where we put $S < S'$ if and only if S is a proper subset of S'. Conversely, if P is any partial order, then P is similar to a partial order P_1 defined as above by means of some family of sets. To show this, let us define, for any a in P, the set $S(a)$ as consisting of a and all x in P such that $x < a$. It is easily verified that $\mathfrak{F} \equiv \{S(a)\}$, for all a in P, is the required family of sets. Any family $\mathfrak{F}$ of sets which defines (in the sense of set inclusion) a partial order similar to a given partial order P will be called a *representation* of P.

3.3. A linear extension L of P will be called *separating* if and only if there exist three elements a, b and c in P such that $a < c$, b is not comparable with either a or c in P, while in L we have $a < b < c$.

3.4. If P is a partial order, then the partial order obtained from P by inverting the sense of all ordered pairs will be denoted by P^*.

3.5. We shall need the following lemma in proving the main theorem of this section.

LEMMA 3.51. *If the partial order P has a conjugate partial order Q, then $A_1 = P + Q$ and $A_2 = P + Q^*$ are both linear extensions of P.*

The proof of this is easy, and will therefore be omitted.

3.6. We shall now prove the following theorem.

THEOREM 3.61. *The following four properties of a partial order P are equivalent.*

(1) *P is reversible.*

(2) *There exists a linear extension of P which is non-separating.*

(3) *The dimension of P is $\leqq 2$.*

(4) *There exists a representation of P by means of a family of intervals on some linearly ordered set.*

Proof. We shall show first that (1) implies (2). Suppose that the partial order P defined on the set S is reversible, and let Q be a partial order on S conjugate to P. By Lemma 3. 51, $A = P + Q$ is a linear extension of P. Let a, b and c be any three elements of S which appear in the order $a < b < c$ in A. If b is not comparable in P with either a or c, then $a < b$ and $b < c$ both appear in Q. Since Q is a partial order, $a < c$ must also appear in Q, and thus a and c are not comparable in P. Hence A is a non-separating linear extension of P.

We show now that (2) implies (3). Suppose that A is a non-separating linear extension of P, and let $Q = A - P$. If $a < b$ and $b < c$ are both in Q, then b is not comparable with either a or c in P. Then $a < c$, which is in A, cannot appear in P, for otherwise $a < b < c$ would be an instance of a separation in A. Hence $a < c$ must also appear in Q, and therefore Q is a partial order. Since Q is conjugate to P, it follows from Lemma 3. 51 that $B = P + Q^*$ is a linear extension of P, and it is obvious that P is realized by the linear orders A and B. Therefore the dimension of P is $\leqq 2$.

We prove next that (3) implies (4). Suppose the dimension of P is $\leqq 2$, and let A and B be any two linear orders on S which together realize P. (If the dimension of P is $= 1$, then P is a linear order L and we may put $A = B = L$). Let B' be a linear order similar to B^*, where the set of elements in B' is disjoint from S. Put $C = B' + A$, so that C is the linear order comprising B' and A, with the additional stipulation that each element of B' precedes each element of A. For each x in P denote by $\bar{x}$ the image of x in the given similarity transformation of B^* into B', and denote by I_x the closed interval $[\bar{x}, x]$ of C. We will show that the family $\{I_x\}$ of all such intervals is a representation of P. Suppose first that $x < y$ in P. Then $x < y$ in A and $\bar{y} < \bar{x}$ in B', so that in C we have $\bar{y} < \bar{x} < x < y$. But this means that I_x is a proper subset of I_y. In the same way it can be shown that if x and y are non-comparable elements of P, then neither of the intervals I_x and I_y contains the other.

To show that (4) implies (1) we shall suppose that P is a partial order (on a set S) which is represented by a family $\{I\}$ of intervals taken from some linear order L. For each x in S, denote by I_x the interval of the family $\{I\}$ which corresponds to x. We notice first that if x and y are distinct elements of S which are not comparable in P, then I_x and I_y cannot have the same left-hand end-point. We define a system Q on S as follows. We put $x < y$ (in Q) if and only if (a) x and y are not comparable in P, and (b) the left-hand end-point of I_x precedes the left-hand end-point of I_y. It is easy to see that Q is a partial order and that Q is conjugate to P. Hence, P is reversible.

3.7. We conclude this section with the following theorem on the question of representation.

THEOREM 3.71. *Let P be a partial order such that if $a < b$ and $a < c$, then b and c are comparable. Then P has a representation in which any two sets are either disjoint or comparable.*

Proof. Let $\mathfrak{F} \equiv \{S(x)\}$ be the representation of P defined in **3.2.** Suppose that x and y are comparable,—say $x < y$. Then clearly $S(x)$ is a proper subset of $S(y)$. If x and y are not comparable, then $S(x) \cdot S(y) = 0$. For suppose the contrary, and let $z \in S(x) \cdot S(y)$. Then $z < x$ and $z < y$. Therefore x and y are comparable, contrary to our supposition.

4. The existence of a partial order having a given dimension.

We first prove the following theorem.

THEOREM 4.1. *For every cardinal number $\mathfrak{n}$ (finite or transfinite), there exists a partial order whose dimension is $\mathfrak{n}$.*

Proof. Let X be any set of elements such that $\bar{\bar{X}} = \mathfrak{n}$. For any x in X, denote by a_x the subset of X whose only element is x, and by c_x the complement of a_x in X. Let $\mathfrak{F}$ denote the family of all sets a_x and c_x, for all x in X, and let P be the partial order *represented* (see **3.2**) by $\mathfrak{F}$. It is clear that, for any two elements a and b of P, we have $a < b$ if and only if there exist x and y in X such that $x \neq y$, $a = a_x$ and $b = c_y$. We shall prove that the dimension of P is $= \mathfrak{n}$ by showing that (1) if $x \neq y$, then no single linear extension of P can contain both $c_x < a_x$ and $c_y < a_y$; and (2) there exist $\mathfrak{n}$ linear extensions of P which realize P.

As to (1), suppose the contrary, and let L be a linear extension of P in which we have both $c_x < a_x$ and $c_y < a_y$. Since $a_x < c_y$ and $a_y < c_x$ in P, we obtain in L: $c_x < a_x < c_y < a_y < c_x$, or $c_x < c_x$, which is impossible.

As to (2), we define L_x, for any x in X, to be any specific linear extension of P in which $c_x < a_x$, and in which $a_y < c_x$ and $a_x < c_y$, for all $y \neq x$. Let $\mathfrak{K} \equiv \{L_x\}$, for all x in X. We have $\bar{\bar{\mathfrak{K}}} = \mathfrak{n}$. Moreover, the set of linear extensions $\mathfrak{K}$ realizes P. Thus, for the non-comparable elements c_x and c_y, $x \neq y$, we have $c_x < a_x < c_y$ in L_x, and $c_y < a_y < c_x$ in L_y, and similarly for the pair a_x and a_y. For the non-comparable elements a_x and c_x we have $c_x < a_x$ in L_x, and $a_x < c_y < a_y < c_x$ in any extension L_y, $y \neq x$. Finally for the comparable elements a_x and c_y, we clearly have $a_x < c_y$ in every extension L_q of $\mathfrak{K}$.

4.2. We proceed to give another example of a partial order of dimension n, for the case in which n is a finite cardinal. We shall need the following lemma.

LEMMA 4.21.[4] *For any permutation θ_n of n distinct natural numbers, there exist k of these numbers which appear in θ_n either in increasing or in decreasing order, where k is the unique natural number such that $(k-1)^2 < n \leqq k^2$.*

Now let n be any natural number and let $p = 2^{2^n} + 1$. Let S be the set whose elements are the first p natural numbers and all pairs (i, j), $i < j$, of these numbers. We define a partial order P on S as follows. If x and y are any two elements of S, let $x < y$ if and only if y is of the form (i, j) and x is either i or j. We now prove the following theorem.

THEOREM 4.22. *The partial order P just defined is of dimension $> n$.*

Proof. Let us assume that the dimension of P is $\leqq n$. There will exist n linear extensions $E_1, E_2, \cdots, E_n$ of P which realize P. The first p natural numbers, as elements of S, appear in a certain permutation in each of these linear extensions. By Lemma 4.21, we can select $2^{2^{n-1}} + 1$ of these numbers which appear monotonically (that is, in numerically increasing or decreasing order) in E_1; from these numbers we can select $2^{2^{n-2}} + 1$ which appear monotonically in E_2, etc.; so that we finally obtain $2^{2^0} + 1 = 3$ numbers which appear monotonically in every one of these linear extensions. Without loss of generality, we may suppose these numbers to be 1, 2 and 3, and that

$$(1) \qquad 1 < 2 < 3 \text{ in } E_i, \qquad (i = 1, 2, \cdots, s);$$

and

$$(2) \qquad 3 < 2 < 1 \text{ in } E_i, \qquad (i = s+1,\ s+2, \cdots, n).$$

Consider now the element $(1, 3)$, which follows both 1 and 3 in P. In each of the first s extensions we will have $2 < 3 < (1, 3)$, and in each of the remaining extensions we will have $2 < 1 < (1, 3)$. Hence in all of the extensions we will have $2 < (1, 3)$, so that $2 < (1, 3)$ in P. But this contradicts the definition of P. It follows, by Theorem 2.33, that there is an integer $q > n$ such that the dimension of P is $= q$.

We can now use P to obtain a partial order whose dimension is n. For let $L_1, L_2, \cdots, L_q$ be linear extensions of P which realize P. It is not hard to show that the partial order P_1 which is realized by $L_1, L_2, \cdots, L_n$, is of dimension n.

[4] This result appears (in slightly different form) in a paper by P. Erdös and G. Szekeres, entitled "A combinatorial problem in geometry," *Compositio Mathematica*, vol. 2 (1935), pp. 463-470.

5. Linear orders in a partial order.

5.1. By a *graph* is meant a set of elements G, together with a binary, symmetric relation $R(x,y)$ which may hold for certain elements x and y of G. If $R(x,y)$ holds, we shall say that x and y are *connected*. A graph is said to be *complete*, if $R(x,y)$ holds for every pair of distinct elements x and y of the graph. It is clear that any partial order gives rise to a graph if we use for $R(x,y)$ the relation "x and y are comparable."

5.2. The theorems on partial orders in this section will be obtained as consequences of certain theorems about graphs. We will prove first the following lemma.

LEMMA 5.21. *If G is a graph of power $\mathfrak{m}$, where $\mathfrak{m}$ is a regular*[5] *cardinal, and if every subset of G of power $\mathfrak{m}$ contains two connected elements, then there exists an element x of G which is connected with $\mathfrak{m}$ elements of G.*

Proof. Assume there is no such element x. Let x_1 be any element of G, and denote by G_1 the set of all elements with which x_1 is connected. We have $\bar{\bar{G}}_1 < \mathfrak{m}$. Let μ denote the initial ordinal such that $\bar{\mu} = \mathfrak{m}$, and suppose that x_α and G_α have been defined and $\bar{\bar{G}}_\alpha < \mathfrak{m}$, for all $\alpha < \beta$, where $\beta < \mu$. Since $\mathfrak{m}$ is regular, $G - \sum_{\alpha<\beta} (x_\alpha + G_\alpha) \neq 0$. Let x_β be any element of $G - \sum_{\alpha<\beta} (x_\alpha + G_\alpha)$. Denote by G_β the set of all elements in G with which x_β is connected. Then $\bar{\bar{G}}_\beta < \mathfrak{m}$. Consider finally the set $X = \sum_{\alpha<\mu} x_\alpha$. No two elements of X are connected, and yet $\bar{\bar{X}} = \mathfrak{m}$. From this contradiction the result follows.

THEOREM 5.22.[6] *If G is a graph of power $\mathfrak{m}$, where $\mathfrak{m}$ is a transfinite cardinal, and if every subset of G of power $\mathfrak{m}$ contains two connected elements, then G contains a complete graph of power $\aleph_0$.*

Proof. We consider first the case where $\mathfrak{m}$ is regular. In virtue of Lemma 5.21, there exists an element x_1 of G which is connected with $\mathfrak{m}$ elements of G. Denote the set of these elements by G_1. We have $\bar{\bar{G}}_1 = \mathfrak{m}$.

[5] For the meaning of the terms *regular* and *singular*, in connection with transfinite numbers, one may refer to Sierpiński's book, *Leçons sur les Nombres Transfinis*. A simple type of example shows that Lemma 5.21 is not true in case $\mathfrak{m}$ is any *singular* cardinal.

[6] We are indebted to P. Erdös for suggestions in connection with Theorems 5.22 and 5.23. In particular, Erdös suggested the proof of 5.22 for the case in which $\mathfrak{m}$ is a singular cardinal.

Suppose now that x_{n-1} and G_{n-1} have been defined and $\bar{\bar{G}}_{n-1} = \mathfrak{m}$. In virtue of Lemma 5.21, there exists an element x_n of G_{n-1} such that (1) $x_n \neq x_k$, for $k < n$, and (2) x_n is connected with $\mathfrak{m}$ elements of G_{n-1}. Denote this set of elements by G_n. Consider finally the set $X = \sum_{n=1}^{\infty} x_n$. It is clear that $\bar{\bar{X}} = \aleph_0$ and that any two elements of X are connected.

We now consider the case where $\mathfrak{m}$ is a singular cardinal. Let $\mathfrak{b}$ denote the smallest cardinal such that $\mathfrak{m}$ is the sum of $\mathfrak{b}$ cardinals each less than $\mathfrak{m}$. Since $\mathfrak{m}$ is singular, we have $\mathfrak{b} < \mathfrak{m}$. Let ϕ denote the initial ordinal such that $\bar{\phi} = \mathfrak{b}$. There will exist regular cardinals $\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_\alpha, \cdots, \alpha < \phi$, such that $\mathfrak{b} < \mathfrak{r}_\alpha < \mathfrak{m}$ and $\mathfrak{m} = \sum_{\alpha < \phi} \mathfrak{r}_\alpha$.

In the first place, if every subset H of G of power $\mathfrak{m}$ contains an element connected with $\mathfrak{m}$ elements of H, then we can proceed, as in the previous case, to obtain a complete graph of power $\aleph_0$. We shall accordingly assume that there exists a subset H of G such that $\bar{\bar{H}} = \mathfrak{m}$ and such that no element of H is connected with $\mathfrak{m}$ elements of H.

We shall show that there exists an $\alpha < \phi$ and a subset Q of H such that $\bar{\bar{Q}} = \mathfrak{r}_\alpha$ and such that every subset of Q of power $\mathfrak{r}_\alpha$ contains two connected elements. Then, by Case 1, there exists in Q, and therefore in G, a complete graph of power $\aleph_0$.

Let us assume the contrary, namely, that there exists no such subset Q of H corresponding to any $\alpha < \phi$. We shall show that this assumption leads to a contradiction.

First, if A is any subset of H, denote by $C(A)$ the set of all elements of H which are connected with the various elements of A. Let K be any subset of H such that $\bar{\bar{K}} = \mathfrak{m}$. Let α be any ordinal $< \phi$. We shall show that K contains a subset W, of power $\mathfrak{r}_\alpha$, with these two properties: (1) no two elements of W are connected, and (2) $\overline{\overline{C(W)}} < \mathfrak{m}$. To prove this, we notice first that by the assumption made in the previous paragraph, there is a subset L of K such that $\bar{\bar{L}} = \mathfrak{r}_\alpha$ and such that no two elements of L are connected. Let L_β denote the set of all x in L such that x is connected with at most $\mathfrak{r}_\beta$ elements of H. We have $L = \sum_{\beta < \phi} L_\beta$. It follows that $\mathfrak{r}_\alpha = \sum_{\beta < \phi} \bar{\bar{L}}_\beta$. Since $\mathfrak{r}_\alpha$ is regular, and $\bar{\phi} = \mathfrak{b} < \mathfrak{r}_\alpha$, we must have $\bar{\bar{L}}_\beta = \mathfrak{r}_\alpha$ for some $\beta < \phi$. We now take W as this set L_β. Clearly (1) no two elements of W are connected, since $W \subset L$, and (2) $\overline{\overline{C(W)}} = \overline{\overline{C(L_\beta)}} \leqq \mathfrak{r}_\alpha \cdot \mathfrak{r}_\beta < \mathfrak{m}$.

To obtain the contradiction we proceed as follows. Denote by W_1 a subset of H such that $\bar{\bar{W}}_1 = \mathfrak{r}_1$, no two elements of W_1 are connected, and $\overline{\overline{C(W_1)}} < \mathfrak{m}$. Suppose we have defined W_α for every $\alpha < \lambda < \phi$, so that $\bar{\bar{W}}_\alpha = \mathfrak{r}_\alpha$, no two elements of W_α are connected, and $\overline{\overline{C(W_\alpha)}} < \mathfrak{m}$. Then $H - \sum_{\alpha<\lambda} \{W_\alpha + C(W_\alpha)\}$ has power $\mathfrak{m}$. (This is the case since ϕ is the *initial* ordinal such that $\bar{\phi} = \mathfrak{b}$). Let W_λ be a subset of $H - \sum_{\alpha<\lambda} \{W_\alpha + C(W_\alpha)\}$ such that $\overline{\overline{W}}_\lambda = \mathfrak{r}_\lambda$, no two elements of W_λ are connected and $\overline{\overline{C(W_\lambda)}} < \mathfrak{m}$. Now consider $\sum_{\lambda<\phi} W_\lambda$. Clearly, this set has power $\mathfrak{m}$ and yet no two elements of it are connected. But this contradicts our hypothesis.

On the basis of the theorem just proved we can now prove the following related theorem.

THEOREM 5.23. *If G is a graph of power $\mathfrak{m}$, where $\mathfrak{m}$ is a transfinite cardinal, and every subset of G of power $\aleph_0$ contains two connected elements, then G contains a complete graph of power $\mathfrak{m}$.*

Proof. Let $R'(x, y)$ mean "x and y are not connected." Let G' denote the graph determined by the elements of the set G in connection with the relation $R'(x, y)$. The application of Theorem 5.22 to the graph G' leads easily to the desired conclusion.

As previously mentioned, a partial order P gives rise to a graph if we let $R(x, y)$ mean "x and y are comparable in P." Hence the two theorems just proved give us the following theorems as corollaries.

THEOREM 5.24. *If P is a partial order of power $\mathfrak{m}$, where $\mathfrak{m}$ is a transfinite cardinal, and if every subset of P of power $\mathfrak{m}$ contains two comparable elements, then P contains a linear order of power $\aleph_0$.*

THEOREM 5.25. *If P is a partial order of power $\mathfrak{m}$, where $\mathfrak{m}$ is a transfinite cardinal, and if every subset of P of power $\aleph_0$ contains two comparable elements, then P contains a linear order of power $\mathfrak{m}$.*

5.3. The question arises as to whether stronger conclusions can be drawn in Theorems 5.22, 5.23, 5.24 and 5.25. We shall consider only a very special case of this problem; namely, the case in which $\mathfrak{m} = \aleph_1$.

Consider first the following example. Let N be any set of power $\aleph_1$. Let C denote the linear continuum, and W the well-ordered series consisting of all the ordinals of the first and second class. Let f and g denote functions (single-valued and having a single-valued inverse) defined on N and such that $f(N) \subset C$ and $g(N) \subset W$, respectively. We denote by P the (reversible) partial order on N which is realized by the two functions f and g. Now if M is any non-denumerable subset of N, there exists an element x of M such that $f(x)$ is a condensation point of $f(M)$ from both the left and the right. There accordingly exist elements y and z of M such that $f(z) < f(x) < f(y)$, $g(x) < g(y)$, and $g(x) < g(z)$. Hence, x and y are comparable in P, while

x and z are not comparable in P.[7] In other words, the partial order P has the following property: Every subset of P of power $\aleph_1$ contains two comparable elements, and yet P contains no linear order of power $\aleph_1$.

Since P is reversible, it can (by virtue of Theorem 3.61) be represented as a family of intervals on some linear order. Hence, the result just obtained can be given the following form.

THEOREM 5.31. *There exists a non-denumerable family* $\mathfrak{F}$ *of intervals (on a certain linear order* A *of power* $\aleph_1$*) which has the following property. Every non-denumerable sub-family* $\mathfrak{F}'$ *of* $\mathfrak{F}$ *contains two comparable intervals, and yet* $\mathfrak{F}$ *contains no non-denumerable monotonic sub-family.*

A stronger result than that of Theorem 5.31 will be presently obtained. This result will depend upon the following theorem.

THEOREM 5.32. *If the hypothesis of the continuum is true, there exists a non-denumerable set* N *of real numbers which has the following property. If* N_1 *and* N_2 *are any two disjoint non-denumerable subsets of* N*, then* $\phi(N_1) \neq N_2$*, where* ϕ *is any increasing or decreasing function defined on* N_1.

Proof. Let us arrange in a well-ordered series of type Ω all real-valued functions $f(x)$ which (a) are monotonic (non-increasing and non-decreasing) on the linear continuum C, and (b) are such that $E[f(x) = x]$ is nowhere dense on C:

$$f_1, f_2, \cdots, f_\alpha, \cdots, (\alpha < \Omega).$$

For a given $\alpha < \Omega$ there may exist an interval on which $f_\alpha(x)$ is constant. The set of all values assumed on such intervals (for a given α) is at most denumerably infinite. Denote this set of values by D_α.

Let x_1 be any real number, and assume that x_β has been defined for all $\beta < \alpha < \Omega$. We shall show that it is possible to choose x_α so that (1) $x_\alpha \neq x_\beta$ for $\beta < \alpha$, (2) $x_\alpha \neq f_\mu(x_\beta)$ for $\mu < \alpha$ and $\beta < \alpha$, (3) $f_\mu(x_\alpha) \neq x_\alpha$ for $\mu < \alpha$, and (4) $f_\mu(x_\alpha) \neq x_\beta$ or $f_\mu(x_\alpha) \in D_\mu$ for $\mu < \alpha$ and $\beta < \alpha$. That x_α can be so defined may be seen as follows. Conditions (1) and (2) can be realized by avoiding a denumerable set. By virtue of (b), we can realize (3) by avoiding a set of the first category. Finally consider any $\mu < \alpha$ and any $\beta < \alpha$. There is at most one x in $C - D_\mu$ for which $f_\mu(x) = x_\beta$. It follows that, except for a countable set of points, we have $f_\mu(x) \in D_\mu$ or $f_\mu(x) \neq x_\beta$ for all $\mu < \alpha$ and all $\beta < \alpha$. Altogether, then, the set of points which has to

[7] In a similar way it can be shown that if N_1 and N_2 are any two disjoint non-denumerable subsets of N, then there exist elements a_1 and b_1 of N_1, and elements a_2 and b_2 of N_2, such that a_1 and a_2 are comparable in P, while b_1 and b_2 are not comparable in P.

be avoided is of the first category. As such a set cannot exhaust C, it is clear that (1), (2), (3) and (4) can be realized.

We now put $N = \sum_{\alpha < \Omega} x_\alpha$. From (1) it follows that N is non-denumerable. Consider now any fixed $\mu < \Omega$, and any λ such that $\mu < \lambda < \Omega$. From (2), (3) and (4) it can be seen that if $f_\mu(x_\lambda) \epsilon N$, then $f_\mu(x_\lambda) \epsilon D_\mu$. It follows that $N \cdot f_\mu(N)$ is denumerable. Hence, for no $\mu < \Omega$ can we have $f_\mu(N_1) = N_2$, where N_1 and N_2 are disjoint non-denumerable subsets of N. Finally, assume that $\phi(N_1) = N_2$, where ϕ is an increasing or decreasing function defined on N_1. For each x in N_1, we have $\phi(x) \neq x$, and it can be easily shown (by suitably extending the definition of ϕ) that there exists a $\mu < \Omega$ such that f_μ agrees with ϕ on N_1. Our result follows from this contradiction.

The result of the preceding theorem can be expressed by saying that if N_1 and N_2 are disjoint non-denumerable subsets of N, then N_1 cannot be mapped onto N_2 by any order-preserving or order-reversing transformation. It follows of course that if N_1 and N_2 are non-denumerable subsets of N such that $N_1 - N_2$ is non-denumerable, then N_1 cannot be mapped onto N_2, for such a mapping would imply that $N_1 - N_2$ could be mapped onto a non-denumerable subset of N_2. In a previous paper[8] the authors have shown that there exists a non-denumerable subset of the linear continuum which is not similar to any proper subset of itself. We note here that the set N just constructed has the following property: If M is any non-denumerable subset of N, then M is not similar to any proper subset of itself which differs from M in more than a denumerable infinity of points.

We now return to our main purpose, and prove the following theorem.

THEOREM 5.33. *The set N of Theorem 5.32 has the following property. Let $\mathcal{F}$ be any non-denumerable family of intervals on the linearly ordered set N such that no two intervals of $\mathcal{F}$ have an end-point in common. Then $\mathcal{F}$ contains two comparable intervals and two non-comparable intervals.*

Proof. Assume that every two intervals of $\mathcal{F}$ are comparable. Let N_1 denote the set of left-hand end-points and N_2 the set of right-hand end-points of the intervals of $\mathcal{F}$. If $n_1' < n_1''$, then $n_2'' < n_2'$, and we obtain an order-reversing transformation of N_1 into N_2. Similarly, if we assume that no two intervals of $\mathcal{F}$ are comparable, we obtain an order-preserving transformation of N_1 into N_2.

UNIVERSITY OF MICHIGAN.

[8] "Concerning similarity transformations of linearly ordered sets," *Bulletin of the American Mathematical Society*, vol. 46 (1940), pp. 322-326.