d-17 The Čech–Stone Compactification

1. Constructions

In [11] Tychonoff not only proved the ***Tychonoff Product Theorem*** (for powers of the unit interval), he also showed that every ***completely regular*** space of ***weight*** κ can be embedded into the space ${}^{\kappa}[0,1]$. This inspired Čech, in [1], to construct for every completely regular space S a ***compact Hausdorff*** space $\beta(S)$ that contains S as a ***dense subspace*** – a **compactification** of S – and such that every bounded real-valued continuous function on S can be extended to a continuous function on $\beta(S)$. The construction amounts to taking the family $\mathcal{C}$ of all continuous functions from S to $[0,1]$, the corresponding ***diagonal map*** $e = \triangle_{f\in\mathcal{C}} f : S \to [0,1]^{\mathcal{C}}$ and obtaining $\beta(S)$ as the closure of $e[S]$ in $[0,1]^{\mathcal{C}}$. Čech also proved that for any other compactification B of S there is a continuous map $h : \beta(S) \to B$ with $h(x) = x$ for $x \in S$ and $h[\beta(S) \setminus S] = B \setminus S$. From this he deduced that if B is a compactification of S in which ***functionally separated*** subsets in S have disjoint closures in B then the map h is a homeomorphism, whence $B = \beta(S)$.

Somewhat earlier, in [10], M.H. Stone applied his theory of representations of ***Boolean algebra***s to various topological problems. One of the major applications is the construction, using the ring $C^*(X)$ of bounded real-valued continuous functions, of a compactification $\mathcal{X}$ of X with the same extension property as the compactification that Čech would construct. The construction proceeds by taking the Boolean algebra $\mathbb{B}$ generated by all ***cozero sets*** and all ***nowhere dense*** subsets of X. As a first step Stone took the representing space $\mathfrak{S}(\mathbb{B})$ – the ***Stone space*** – of $\mathbb{B}$. Next, to every maximal ideal $\mathfrak{m}$ of $C^*(X)$ he associated the ideal $I_{\mathfrak{m}}$ of $\mathbb{B}$ consisting of those sets B in $\mathbb{B}$ for which there are $f \in C^*(X)$ and $a, b, c \in \mathbb{R}$ such that $A \subseteq f^{\leftarrow}[(a,b)]$; $f \equiv c \pmod{\mathfrak{m}}$ and $c < a$ or $b < c$. Finally $F_{\mathfrak{m}}$ is the closed subset of $\mathfrak{S}(\mathbb{B})$ determined by the filter that is dual to $I_{\mathfrak{m}}$. The space $\mathcal{X}$ is the quotient space of $\mathfrak{S}(\mathbb{B})$ by the decomposition consisting of the sets $F_{\mathfrak{m}}$. One obtains an embedding of X into $\mathcal{X}$ by associating x with the maximal ideal $\mathfrak{m}_x = \{f : f(x) = 0\}$. Stone also proved that every continuous map on X with a compact co-domain can be extended to $\mathcal{X}$.

The compactification constructed by Čech and Stone is nowadays called the **Čech–Stone compactification**;[1] Čech's β is still used, we write βX (without Čech's parentheses). The properties of βX that Čech and Stone established each characterize it among all compactifications of X: (1) it is the maximal compactification; (2) functionally separated subsets of X have disjoint closures in βX; and (3) every continuous map from X to a compact space extends to all of βX (the extension of $f : X \to K$ is denoted βf). Of the many other constructions of βX that have been devised we mention two. First, in [6], Gel'fand and Kolmogoroff showed that the ***hull-kernel topology*** on the set of maximal ideals of $C^*(X)$ immediately gives us βX and that one may also use the ring $C(X)$ of all real-valued continuous functions on X. Second, in [7], Gillman and Jerison gave what for many is the definitive construction of βX: the ***Wallman compactification*** of X with respect to the family $\mathcal{Z}(X)$ of all ***zero set***s of X. This means βX is the set of all ultrafilters on the family $\mathcal{Z}(X)$ – the z**-ultrafilter**s or **zero-set ultrafilters** – with the family $\{\overline{Z} : Z \in \mathcal{Z}(X)\}$ as a base for the closed sets, where $\overline{Z} = \{u \in \beta X : Z \in u\}$. One identifies a point x of X with the z-ultrafilter $u_x = \{Z : x \in Z\}$.

Property 2 above is usually reformulated as: (2′) disjoint zero sets of X have disjoint closures in βX. For ***normal spaces*** one can obtain βX as *the* Wallman compactification of X, i.e., using the family of all closed sets; property (2′) then becomes (2″) disjoint closed sets in X have disjoint closures in βX. The equality $\beta X = K$ should be taken to mean that K is compact and there is an embedding $f : X \to K$, for which $f[X]$ is dense in K and for which the extension βf is a homeomorphism – especially if X is dense in K. In this sense the notation βf is unambiguous: the graph of βf is the Čech–Stone compactification of the graph of f.

The assignment $X \mapsto \beta X$ is a ***covariant functor*** from the ***category*** of Tychonoff spaces to the category of compact Hausdorff spaces, both with continuous maps as **morphisms**. It is in fact the adjoint of the ***forgetful functor*** from compact Hausdorff spaces to Tychonoff spaces. This gives another way of proving that "βX exists": because the category of compact Hausdorff spaces is closed under products and closed subsets. In fact Čech's construction of βX may be construed as a forerunner of the Adjoint Functor Theorem.

2. Properties

We say that a subspace A is C**-embedded** (C^***-embedded**) in a space X if every (bounded) real-valued continuous function on A can be extended to a continuous real-valued function on X. Thus a completely regular space X is C^*-embedded in its Čech–Stone compactification βX and any compactification of X in which X is C^*-embedded must be βX. These remarks help us to recognize some Čech–Stone compactifications: if $A \subseteq X$ then $\mathrm{cl}_{\beta X} A = \beta A$ iff A is C^*-embedded in X and $\beta Y = \beta X$ whenever $X \subseteq Y \subseteq \beta X$. If X is normal then $\mathrm{cl}_{\beta X} A = \beta A$ whenever A is closed in X (by the ***Tietze–Urysohn theorem***).

[1] In Europe; elsewhere one speaks of the **Stone–Čech compactification**.

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One can use C^*-embedding to calculate βX explicitly for a few X, by which we mean that βX is an already familiar space. The well-known fact that every continuous real-valued function on the ***ordinal space*** ω_1 is constant on a tail implies that $\beta\omega_1 = \omega_1 + 1$ – the same holds for every ordinal of uncountable cofinality. Other examples are provided by ***Σ-products***: if κ is uncountable and $X = \{x\colon \{\alpha\colon x_\alpha \neq 0\}$ is countable$\}$ as a subspace of ${}^\kappa[0,1]$ (or ${}^\kappa 2$) then every continuous real-valued function on X depends on countably many coordinates and hence can be extended to the ambient product so that $\beta X = {}^\kappa[0,1]$ (or $\beta X = {}^\kappa 2$). Note that these spaces, with an easily identifiable Čech–Stone compactification, are ***pseudocompact***. In fact if X is not pseudocompact then it contains a C-embedded copy of $\mathbb{N}$, whence βX contains a copy of $\beta\mathbb{N}$. The space $\beta\mathbb{N}$ is, in essence, hard to describe: if u is a point in $\beta\mathbb{N} \setminus \mathbb{N}$ and so a free ***ultrafilter*** on $\mathbb{N}$ then the set $\{\sum_{n\in U} 2^{-n}\colon U \in u\}$ is a non-Lebesgue measurable set of reals. This illustrates that the construction of βX requires a certain amount of Choice, indeed, the existence of βX is equivalent to the Tychonoff Product Theorem for compact Hausdorff spaces, which, in turn, is equivalent to the Boolean Prime Ideal Theorem.

The map $f \mapsto \beta f$ from $C^*(X)$ to $C(\beta X)$ is an isomorphism of rings (or lattices, or Banach spaces …); this explains why very often investigations into $C^*(X)$ assume that X is compact; this gives the advantage that ideals are fixed, i.e., if I is an ideal of $C^*(X)$ then there is a point x with $f(x) = 0$ for all $f \in I$. Furthermore the maximal ideals are precisely the ideals of the form $\{f\colon f(x) = 0\}$ for some x.

A **perfect map** is one which is continuous, closed and with compact fibers. A map $f: X \to Y$ between completely regular spaces is perfect iff its Čech-extension βf satisfies $\beta f[\beta X \setminus X] \subseteq \beta Y \setminus Y$ or equivalently $X = \beta f^{-1}[Y]$. One can use this, for instance, to show that complete metrizability is preserved by perfect maps (also inversely if the domain is metrizable). A metrizable space is ***completely metrizable*** iff it is a ***G_δ-set*** in its Čech–Stone compactification; the latter property is then easily seen to be preserved both ways by perfect maps. One calls a space a **Čech-complete space** (sometimes **topologically complete**) if it is a G_δ-set in its Čech–Stone compactification. This is an example of βX providing a natural setting for defining or characterizing topological properties of X – the best-known example being of course local compactness: a space is ***locally compact*** iff it is open in its Čech–Stone compactification. Other properties that can be characterized via βX are: the ***Lindelöf*** property (X is **normally placed** in βX, which means that for every open set $U \supseteq X$ there is an F_σ-set F with $X \subseteq F \subseteq U$) and ***paracompactness*** ($X \times \beta X$ is normal).

The product $\beta\mathbb{N} \times \beta\mathbb{N}$ is *not* $\beta(\mathbb{N} \times \mathbb{N})$: the characteristic function of the diagonal of $\mathbb{N}$ witnesses that $\mathbb{N} \times \mathbb{N}$ is not C^*-embedded in $\beta\mathbb{N} \times \beta\mathbb{N}$. The definitive answer to the question when $\beta\prod = \prod\beta$ was given in [8]: if both X and Y are infinite then $\beta X \times \beta Y = \beta(X \times Y)$ iff $X \times Y$ is pseudocompact and the same holds for arbitrary products, with a similar proviso: $\prod_i \beta X_i = \beta\prod_i X_i$ iff $\prod_i X_i$ is pseudocompact, provided $\prod_{i\neq i_0} X_i$ is never finite. If the product can be factored into two subproducts without isolated points then even the homeomorphy of $\prod_i \beta X_i$ and $\beta\prod_i X_i$ implies $\prod_i X_i$ is pseudocompact.

If X is not compact then no point of $X^* = \beta X \setminus X$ is a G_δ-set, in fact a G_δ-set of βX that is a subset of X^* contains a copy of $\mathbb{N}^*$. This implies that nice properties like metrizability, and second- or first-countability do not carry over to βX. Of course separability carries over from X to βX, but not conversely: a Σ-product in ${}^{\mathfrak{c}}2$ is not separable but ${}^{\mathfrak{c}}2$ is.

Properties that are carried over both ways are usually of a global nature. Examples are ***connectedness***, ***extremal disconnectedness***, ***basic disconnectedness*** and the values of the ***large inductive dimension*** (for normal spaces) and ***covering dimension***. These properties have in common that they can be formulated using the families of (co)zero sets and/or the ring $C^*(X)$, which makes it almost automatic for each that X satisfies it iff βX does. Interestingly βX is ***locally connected*** iff X is locally connected and pseudocompact; so, e.g., $\beta\mathbb{R}$ is connected but not locally connected.

A particularly interesting class of spaces in this context is that of the **F-space**s; it can be defined topologically (cozero sets are C^*-embedded) or algebraically (every finitely generated ideal in $C^*(X)$ is principal). The algebraic formulation shows that X is an F-space iff βX is, because $C(\beta X)$ and $C^*(X)$ are isomorphic. Also, X^* is an F-space whenever X is locally compact and σ-compact. Neither property by itself guarantees that X^* is F-space: $\mathbb{Q}^*$ is not an F-space, nor is $(\omega_1 \times [0,1))^*$ (which happens to be $[0,1]$).

This result shows that F-spaces are quite ubiquitous; for example, $\mathbb{N}^*$, $\mathbb{R}^*$, and $(\mathbb{R}^n)^*$ are F-spaces, as well as $(\bigoplus_n X_n)^*$ for any topological sum of countably many compact spaces. An F-space imposes some rigidity on maps having it for its range: if $f: X^\kappa \to Z$ is a continuous map from a power of the compact space X to an F-space Z then X^κ can be covered by finitely many clopen sets such that f depends on one coordinate on each of them. This implies, e.g., that a continuous map from a power of $[0,\infty)^*$ to $[0,\infty)^*$ itself depends on one coordinate only.

Some of the properties mentioned above have relationships beyond the implications between them. Every ***P-space*** is basically disconnected. But an extremally disconnected P-space that is not of a ***measurable cardinal*** number is discrete (the converse is clearly also true). As noted above X is extremally (or basically) disconnected iff βX is but there is more: if X is extremally disconnected or a P-space then βX can be embedded into βD for a large discrete space D. Every compact subset of βD is an F-space but a characterization of the compact subspaces of βD is not known. In the special case of $\beta\mathbb{N}$ there is a characterization under the assumption of the ***Continuum Hypothesis***, but it is also consistent that not all basically disconnected spaces embed into $\beta\mathbb{N}$ and that not every F-space embeds into a basically disconnected space.

As seen above, pseudocompactness is a property that helps give positive structural results about βX; this happens again in the context of topological groups. If X is a topological group then the operations can be extended to βX, making βX into a topological group, if and only if X is pseudocompact.

3. Special points

It is a general theorem that $X^* = \beta X \setminus X$ is not ***homogeneous*** whenever X is not pseudocompact [5]. This in itself very satisfactory result prompted further investigation into the structure of remainders and a search for more reasons for this nonhomogeneity. Many special points were defined that would exhibit different topological behaviour in X^* or βX. The best known are the remote points: a point p of X^* is a **remote point** of X if $p \notin \mathrm{cl}_{\beta X} N$ for all nowhere dense subsets N of X. If p is a remote point of X then βX is extremally disconnected at p.

Many spaces have remote points, e.g., spaces of countable π***-weight*** (or even with a σ-locally finite π***-base***) and $\omega \times {}^{\kappa}2$. If X is **nowhere locally compact**, i.e., when X^* is dense in βX, then X^* is ***extremally disconnected*** at every remote point of X. This gives another reason for the nonhomogeneity of, for example, $\mathbb{Q}^*$, as this space is not extremally disconnected.

Under CH all separable spaces have remote points but in the side-by-side Sacks model there is a separable space without remote points. Many spaces with the ***countable chain condition*** have remote points and it is unknown whether there is such a space without remote points. Proofs that certain spaces have remote points have generated interesting combinatorics; the proof for $\omega \times {}^{\kappa}2$ contains a crucial ingredient for one proof of the consistency of the ***Normal Moore Space Conjecture***.

Further types of points are obtained by varying on the theme of ‘not in the closure of a small set’. Thus one obtains **far point**s: not in the closure of any closed discrete subset of X; requiring this only for countable discrete sets defines ω**-far points** – a **near point** is a point that is not far.

Of earlier vintage are P**-point**s, points for which the family of neighbourhoods is closed under countable intersections. They occured in the algebraic context: a point is a P-point iff every continuous function is constant on a neighbourhood of it. This means that for a P-point x the ideals $\{f\colon f(x) = 0\}$ and $\{f\colon x \in \mathrm{int}\, f^{\leftarrow}(f(x))\}$ coincide. Under CH or even MA one can prove that many spaces of the form X^* have P-points, thus obtaining witnesses to the nonhomogeneity of X^*. Many of these results turned out to be independent of ZFC. The search for a general theorem that would, once and for all, establish nonhomogeneity of X^* by means of a ‘simple’ topological property of some-but-not-all points lead to weak P-points. A **weak** P**-point** is one that is not an accumulation point of any countable set. Their advantage over P-points is that their existence is provable in ZFC, first for $\mathbb{N}^*$, later for more spaces. The final word has not been said, however. The weakest property that has not been ruled out by counterexamples is ‘not an accumulation point of a countable discrete set’.

Further reading

Walker’s book [12] gives a good survey of work on βX up to the mid 1970s. Van Douwen’s [4, 2, 3] and van Mill’s [9] laid the foundations for the work on βX in more recent years.

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Alan Dow and Klaas Pieter Hart

Charlotte, NC, USA and Delft, The Netherlands