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The proposition that a decreasing sequence of ordinals necessarily terminates has been given a new, and perhaps unexpected, importance by the rôle which it plays in Gentzen's proof[1] of the freedom from contradiction of the "reine Zahlentheorie." Gödel's construction[2] of non-demonstrable propositions and the establishment of the impossibility of a proof of freedom from contradiction, within the framework of a certain type of formal system, showed that a proof of freedom from contradiction could be found only by transcending the axioms and proof processes of that formal system. Gentzen's proof succeeds by utilising *transfinite induction* to prove that certain sequences of reduction processes, enumerated by ordinals less than ϵ (the first ordinal to satisfy $\epsilon = \omega^\epsilon$) are finite. Were it possible to prove *the restricted ordinal theorem*, that a descending sequence of ordinals, less than ϵ, is finite, in Gentzen's "reine Zahlentheorie," then it would be possible to determine a contradiction in that number system. In his paper, Gentzen proves the theorem of transfinite induction, which he requires, by an intuitive argument. There is also a method of reducing transfinite induction, for ordinals less than ϵ, to a number-theoretic principle given by Hilbert and Bernays,[3] and a similar method by Ackermann.[4] None of these proofs of transfinite induction is *finitist.*

As the restricted ordinal theorem is a suggested minimum deviation from the previously accepted field of finitist processes, it becomes highly important to examine to what extent this theorem fulfils general finitist requirements. For this purpose it is necessary to give an account of the ordinal signs which does not presuppose any part of the Cantor theory of infinite classes, and in fact such an account is given in Gentzen's paper, but it is more convenient for our purpose to present the construction of ordinal signs differently from Gentzen.

By means of additions, multiplications, and exponentiations we can express any numeral n uniquely in the form

$$c_k s^{a_k} + c_{k-1} s^{a_{k-1}} + \cdots + c_2 s^{a_2} + c_1 s^{a_1} + c_0,$$

where $s \geqq 2, 0 \leqq c_0 < s, 0 < c_1, c_2, c_3, \cdots, c_k < s, 0 < a_1 < a_2 < a_3 < \cdots < a_k$, and each a_i is itself of this form. We shall call this the representation of n with digits $0, 1, 2, \cdots, s-1$ and scale symbol s. If $\phi_s(n)$ is an *abbreviation* for the representation of n with scale symbol s, then this expression may be defined recursively as being the same as $cs^{\phi_s(a)} + \phi_s(n - cs^a)$, where a is the exponent

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[1] G. Gentzen, *Die Widerspruchsfreiheit der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 112 (1936), pp. 493–565.

[2] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, **Monatshefte für Mathematik und Physik,** vol. 38 (1931), pp. 173–198.

[3] D. Hilbert and P. Bernays, **Grundlagen der Mathematik,** vol. II, Berlin 1939, pp. 360–372.

[4] W. Ackermann, *Zur Widerspruchsfreiheit der Zahlentheorie*, **Mathematische Annalen,** vol. 117 (1940), pp. 162–194.

of the greatest power of s which does not exceed n, and cs^a is the greatest multiple of s^a not exceeding n.

Denoting by $\mathrm{Sb}_y^x\alpha$ the expression obtained by replacing x, at each point of its occurrence in an expression α, by y, we define $\mathrm{T}_x^m(n) = \mathrm{Sb}_x^m\phi_m(n)$, the operator Sb_x^m applying to the expression for which $\phi_m(n)$ stands, not just to the sign $\phi_m(n)$ itself (so that $\mathrm{Sb}_x^m\phi_m(n)$ is not $\phi_x(n)$), and define the ordinals (less than ϵ) to be the expressions $\mathrm{T}_\omega^m(n)$ for any m and n, $m \geqq 2$. Thus for instance $\mathrm{T}_\omega^3(106)$ is the ordinal $\omega^{\omega+1} + 2\omega^2 + 2\omega + 1$ (express 106 in the scale of 3 with digits 0, 1, 2 and then replace "3" by "ω"). Every ordinal α, less than ϵ, in the Cantor theory, is expressible in the form $\mathrm{T}_\omega^m(n)$, m being any natural number greater than each of the natural numbers which occur as coefficients or exponents in the expression of α by powers of ω and sums of such powers with numerical coefficients, and n being uniquely determined by α and m.

We shall also use the sign $\mathrm{S}_y^x(n)$, with natural numbers x, y, n, where $y \geqq x > 1$, to denote the number obtained by substituting "y" for "x" in the expression representing n in the scale of x; i.e., $\mathrm{S}_y^x(n)$ is the number which is represented by $\mathrm{Sb}_y^x\phi_x(n)$ in the scale of y. For example $\mathrm{S}_4^3(34) = 265$, since $34 = 3^3 + 2\cdot 3 + 1$ and $4^4 + 2\cdot 4 + 1 = 265$; and $\mathrm{S}_4^2(16) = 4^{256}$, since $16 = 2^{2^2}$ and $4^{4^4} = 4^{256}$.

The formulae $\mathrm{T}_\omega^m(n)$ are not all distinct, for we can show that corresponding to any $m' > m$ we can find n' such that $\mathrm{T}_\omega^{m'}(n')$ and $\mathrm{T}_\omega^m(n)$ are the same formula; in fact if $n' = \mathrm{S}_{m'}^m(n)$ then $\mathrm{T}_\omega^{m'}(n') = \mathrm{T}_\omega^m(n)$, for by definition $\mathrm{Sb}_{m'}^m\phi_m(n) = \phi_{m'}(\mathrm{S}_{m'}^m(n))$ and therefore $\mathrm{T}_\omega^{m'}(\mathrm{S}_{m'}^m(n)) = \mathrm{Sb}_\omega^{m'}\phi_{m'}(\mathrm{S}_{m'}^m(n)) = \mathrm{Sb}_\omega^{m'}\mathrm{Sb}_{m'}^m\phi_m(n) = \mathrm{Sb}_\omega^m\phi_m(n) = \mathrm{T}_\omega^m(n)$.

For any $n_1, n_2, m_1 \geqq m_2 > 1$ we say that $\mathrm{T}_\omega^{m_1}(n_1)$ is greater than, equal to, or less than $\mathrm{T}_\omega^{m_2}(n_2)$ according as n_1 is greater than, equal to, or less than $\mathrm{S}_{m_1}^{m_2}(n_2)$; this definition is in accordance with the usual definition of inequalities between ordinals. A decreasing sequence of ordinals takes the form

$$\mathrm{T}_\omega^{m_1}(n_1),\ \mathrm{T}_\omega^{m_2}(n_2),\ \mathrm{T}_\omega^{m_3}(n_3),\ \cdots,\ \mathrm{T}_\omega^{m_r}(n_r),\ \cdots,$$

where, for each value of r, $m_{r+1} \geqq m_r$ and $n_{r+1} < \mathrm{S}_{m_{r+1}}^{m_r}(n_r)$. For every constructively given sequence of ordinals the sequence m_r is general recursive though not perhaps primitive recursive in every case. For a given function m_r we obtain the 'longest' sequence by taking $n_{r+1} = \mathrm{S}_{m_{r+1}}^{m_r}(n_r) \dot{-} 1$, for $\mathrm{T}_\omega^m(n) = 0$ if and only if $n = 0$, and if $n < n_r$ then $\mathrm{S}_{m_{r+1}}^{m_r}(n) < \mathrm{S}_{m_{r+1}}^{m_r}(n_r)$.

The restricted ordinal theorem may now be expressed by saying that for any non-decreasing function p_r, $p_0 \geqq 2$, and for n_r defined by the recursive equation $n_{r+1} = \mathrm{S}_{p_{r+1}}^{p_r}(n_r) \dot{-} 1$, we can find a value of r for which $\mathrm{T}_\omega^{p_r}(n_r) = 0$.

We observe first that the restricted ordinal theorem is equivalent to the following *number-theoretic proposition*:

Given any non-decreasing function p_r, $p_0 \geqq 2$, a number n_0, and the function n_r defined by the recursive equation $n_{r+1} = \mathrm{S}_{p_{r+1}}^{p_r}(n_r) \dot{-} 1$, then there is a value of r for which $n_r = 0$.

We shall call this proposition P^*. It makes no essential difference in forming the sequence $n_0, n_1, n_2, \cdots$ whether in forming n_{r+1} from n_r we first reduce n_r by unity and then change the scale, or as we have done above, first change the scale in the representation of n_r and then reduce the resulting number by unity.

In fact if we form a sequence m_r by the recursive equation $m_{r+1} = \mathrm{S}^{p_r}_{p_{r+1}}(m_r \dot{-} 1)$, then the proposition P^* above is proved if we can prove the proposition P that there is a value of r for which $m_r = 0$. For if $m_r > 0$, for $r \leqq s$, and $m_{s+1} = 0$, then taking $n_0 = m_0 \dot{-} 1$, from $n_k = m_k \dot{-} 1$ we derive $n_{k+1} = \mathrm{S}^{p_k}_{p_{k+1}}(n_k) \dot{-} 1 = \mathrm{S}^{p_k}_{p_{k+1}}(m_k \dot{-} 1) \dot{-} 1 = m_{k+1} \dot{-} 1$, and therefore $n_r = m_r \dot{-} 1$, for all r, whence $n_{s+1} = 0$.

We shall give a completely finitist proof of the proposition P (constructing an explicit formula determining a value of r for which $m_r = 0$) for values of m_0 not greater than $p_0^{p_0^{p_0}}$. This is equivalent to proving the restricted ordinal theorem for ordinals not greater than $\omega^{\omega^{\omega}}$.

It will make the demonstration easier to follow if we consider first the case $m_0 \leqq p_0^{p_0}$.

Let $\sigma(n)$ be a non-decreasing sequence, $\sigma(0) \geqq 2$, and let a sequence $\gamma_\sigma(x, n, p, r)$ be defined by the equations:

$$\gamma_\sigma(x, p, n, 0) = x\{\sigma(n)\}^p,$$

$$\gamma_\sigma(x, p, n, r + 1) = \mathrm{S}^{\sigma(n+r)}_{\sigma(n+r+1)}\{\gamma_\sigma(x, p, n, r) \dot{-} 1\}.$$

Define the function $f_\sigma(x, p, n)$ by the equations:

$$\phi_\sigma(a, b, c, n) = c + f_\sigma(a, b, n + c),$$

$$f_\sigma(1, 0, n) = 1, \tag{i}$$

$$f_\sigma(x + 2, p, n) = \phi_\sigma(x + 1, p, f_\sigma(1, p, n), n), \tag{ii}$$

$$f_\sigma(1, p + 1, n) = \phi_\sigma(\sigma(n) \dot{-} 1, p, f_\sigma(1, p, n), n), \tag{iii}$$

where $x \geqq 0, p \geqq 0, n \geqq 0$. Then for all $x + 1, p < \sigma(n)$, $k \geqq f_\sigma(x + 1, p, n)$, $\gamma_\sigma(x + 1, p, n, k) = 0$.

For x, p, $n \geqq 0$ let $P_\sigma(x + 1, p, n)$ denote the proposition, "If $k = f_\sigma(x + 1, p, n)$ and $x + 1, p < \sigma(n)$ then $\gamma_\sigma(x + 1, p, n, k) = 0$." Equation (i) proves $P_\sigma(1, 0, n)$. And equation (ii) proves $P_\sigma(1, p, n)$ & $P_\sigma(x + 1, p, n + f_\sigma(1, p, n)) \rightarrow P_\sigma(x + 2, p, n)$; for starting from $(x + 2)\{\sigma(n)\}^p$, with $x + 2 < \sigma(n)$, $p < \sigma(n)$, we reach in turn $(x + 1)\{\sigma(n + 1)\}^p + \mathrm{S}^{\sigma(n)}_{\sigma(n+1)}[\{\sigma(n)\}^p \dot{-} 1]$, and $(x + 1)\{\sigma(n + 2)\}^p + \mathrm{S}^{\sigma(n+1)}_{\sigma(n+2)}[\mathrm{S}^{\sigma(n)}_{\sigma(n+1)}[\{\sigma(n)\}^p \dot{-} 1] \dot{-} 1]$, and so on up to $(x + 1)\{\sigma(n + f_\sigma(1, p, n))\}^p$ in $f_\sigma(1, p, n)$ steps, and $(x + 1)\{\sigma(n + f_\sigma(1, p, n))\}^p$ is reduced to zero in a further $f_\sigma(x + 1, p, n + f_\sigma(1, p, n))$ steps. Furthermore, starting from $\{\sigma(n)\}^{p+1}$, where $p + 1 < \sigma(n)$, the next term is $\{\sigma(n) \dot{-} 1\}\{\sigma(n + 1)\}^p + \mathrm{S}^{\sigma(n)}_{\sigma(n+1)}[\{\sigma(n)\}^p \dot{-} 1]$, and so on, so that equation (iii) proves $P_\sigma(1, p, n)$ & $P_\sigma(\sigma(n) \dot{-} 1, p, n + f_\sigma(1, p, n)) \rightarrow P_\sigma(1, p + 1, n)$.

From the proved propositions,

$$P_\sigma(1, 0, n), \tag{a}$$

$$P_\sigma(1, p, n) \;\&\; P_\sigma(x + 1, p, n + f_\sigma(1, p, n)) \rightarrow P_\sigma(x + 2, p, n), \tag{b}$$

$$P_\sigma(1, p, n) \;\&\; P_\sigma(\sigma(n) \dot{-} 1, p, n + f_\sigma(1, p, n)) \rightarrow P_\sigma(1, p + 1, n), \tag{c}$$

we can derive $P_\sigma(x + 1, p, n)$ by an application of the generalised schema of induction II described in Th. Skolem's paper *Eine Bemerkung über die Induk-*

tionsschemata in der rekursiven Zahlentheorie,[5] which, as Skolem shows, if we take into account the observation which Miss R. Péter makes in her review of Skolem's paper,[6] can be reduced to an ordinary induction. For by generalised induction the formula $P_\sigma(x+1, p, n)$, with variables x, n and some definite numeral p, is derived from $P_\sigma(1, p, n)$, with the same numeral p, by means of formula (b) above; in particular $P_\sigma(\sigma(n) \dot{-} 1, p, n + f_\sigma(1, p, n))$ is derivable from $P_\sigma(1, p, n)$ and hence by (c) we derive $P_\sigma(1, p+1, n)$ from $P_\sigma(1, p, n)$. From this, in conjunction with (a), we then derive $P_\sigma(1, p, n)$ by induction over p, from which we conclude that $P_\sigma(x+1, p, n)$ holds for arbitrary values of $x, p, n \geqq 0$.

This is a *finite constructive* proof of the restricted ordinal theorem for ordinals less than ω^ω.

Next we observe that, writing R for $\sigma(n) \dot{-} 1$, we have $\{\sigma(n)\}^{\sigma(n)} = R\{\sigma(n)\}^R + \{\sigma(n)\}^R$, and therefore the sequence $\delta_\sigma(n, r)$, with $\delta_\sigma(n, 0) = \{\sigma(n)\}^{\sigma(n)}$ and $\delta_\sigma(n, r+1) = S^{\sigma(n+r)}_{\sigma(n+r+1)}\{\delta_\sigma(n, r) \dot{-} 1\}$, reaches zero in

$$f_\sigma(1, R, n) + f_\sigma(R, R, n + f_\sigma(1, R, n)) = f_\sigma(1, \sigma(n), n)$$

steps, for $f_\sigma(1, R, n)$ steps take us from $R\{\sigma(n)\}^R + \{\sigma(n)\}^R$ to $R\{\sigma(n + f_\sigma(1, R, n))\}^R$, and therefore a further $f_\sigma(R, R, n + f_\sigma(1, R, n))$ steps are needed to reach zero. Thus the restricted ordinal theorem is proved for ordinals less than or equal to ω^ω. (Notice that the formula $f_\sigma(1, \sigma(n), n)$ for the number of terms in a sequence commencing with $\{\sigma(n)\}^{\sigma(n)}$ is the same as the formula for a sequence commencing with $\{\sigma(n)\}^p$, $p < n$, with p replaced by $\sigma(n)$; this is to be expected since the relation of $\{\sigma(n)\}^{\sigma(n)}$ to $\{\sigma(n)\}^R$ is the same as the relation of $\{\sigma(n)\}^R$ to $\{\sigma(n)\}^{R \dot{-} 1}$.)

Consider next the sequence $\epsilon_\sigma(x, y_0, y_1, \ldots, y_j, n, r)$ with

$$\epsilon_\sigma(x, y_0, y_1, \cdots, y_j, n, 0) = (x+1)\{\sigma(n)\}^{y_0 + y_1\sigma(n) + y_2[\sigma(n)]^2 + \cdots + y_j[\sigma(n)]^j}$$

and

$$\epsilon_\sigma(x, y_0, y_1, \cdots, y_j, n, r+1) = S^{\sigma(n+r)}_{\sigma(n+r+1)}\{\epsilon_\sigma(x, y_0, y_1, \cdots, y_j, n, r) \dot{-} 1\}.$$

The function $f_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n)$ is defined by the equations (recursive for a definite value of j):

$$f_{\sigma,j}(0, 0, \cdots, 0, n) = 1, \tag{iv}$$

$$f_{\sigma,j}(x+1, y_0, y_1, \cdots, y_j, n) = \phi_{\sigma,j}(x, y_0, y_1, \cdots, y_j, f_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n), n), \tag{v}$$

$$f_{\sigma,j}(0, 0, \cdots, 0, y_r + 1, y_{r+1}, y_{r+2}, \cdots, y_j, n) = \phi_{\sigma,j}(R \dot{-} 1, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, f_{\sigma,j}(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n), n), \tag{vi}$$

where $j, x \geqq 0$, $0 \leqq r \leqq j$, $R = \sigma(n) \dot{-} 1$, and $\phi_{\sigma,j}(x, y_0, y_1, \cdots, y_j, c, n) = c + f_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n + c)$.

[5] ***Monatshefte für Mathematik und Physik,*** vol. 48 (1939), pp. 268–276.

[6] In this JOURNAL, vol. 5 (1940), pp. 34–35.

Let $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n)$ say, "If $k = f_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n)$ and $x, y_0, y_1, \cdots, y_j < \sigma(n)$ then $\epsilon_\sigma(x, y_0, y_1, \cdots, y_j, n, k) = 0$."

Equation (iv) proves $P_{\sigma,j}(0, 0, \cdots, 0, n)$. Since $(x + 2)\{\sigma(n)\}^s = (x + 1)\{\sigma(n)\}^s + \{\sigma(n)\}^s$, equation (v) proves

$$\{P_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n) \,\&\, P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n + f_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n)\} \rightarrow P_{\sigma,j}(x + 1, y_0, y_1, \cdots, y_j, n). \tag{g}$$

And since

$$\{\sigma(n)\}^{(y_r+1)[\sigma(n)]^r + y_{r+1}[\sigma(n)]^{r+1} + \cdots + y_j[\sigma(n)]^j} = R\{\sigma(n)\}^{R+R\sigma(n)+\cdots+R[\sigma(n)]^{r-1}+y_r[\sigma(n)]^r+\cdots+y_j[\sigma(n)]^j} + \{\sigma(n)\}^{R+R\sigma(n)+\cdots+R[\sigma(n)]^{r-1}+y_r[\sigma(n)]^r+\cdots+y_j[\sigma(n)]^j},$$

therefore equation (vi) proves

$$\{P_{\sigma,j}\{0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n) \,\&\, P_{\sigma,j}(R \dot{-} 1, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n + f_{\sigma,j}(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n))\} \rightarrow P_{\sigma,j}(0, 0, \cdots, 0, y_r + 1, y_{r+1}, y_{r+2}, \cdots, y_j, n). \tag{h}$$

If for given values of $x, y_0, y_1, \cdots, y_j$ we can derive, for any assigned m, $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m)$ from $P_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n)$ utilising only the elementary propositional calculus and the operations of substituting for variables and replacing computable functional expressions by their values, then in particular we can derive $P_{\sigma,j}(0, y_0, y_1, \cdots, y_j, m)$ and $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m + f_{\sigma,j}(0, y_0, y_1, \cdots, y_j, m))$ by these means, and hence by (g) we derive $P_{\sigma,j}(x + 1, y_0, y_1, \cdots, y_j, m)$, whence it follows, by induction over x, that we can derive $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m)$ from $P_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n)$, for any x. Furthermore, if for given values of $x, r, y_r, y_{r+1}, \cdots, y_j$ we can derive $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m)$ for any assigned m, from $P_{\sigma,j}(0, 0, \cdots, 0, y_r, y_{r+1}, \cdots, y_j, n)$, then we can derive both $P_{\sigma,j}(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, m)$ and $P_{\sigma,j}(R \dot{-} 1, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, m + f_{\sigma,j}(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, m))$ and hence by (h), $P_{\sigma,j}(0, 0, \cdots, 0, y_r + 1, y_{r+1}, y_{r+2}, \cdots, y_j, m)$. By induction over y_r it follows that from $P_{\sigma,j}(0, 0, \cdots, 0, 0, y_{r+1}, y_{r+2}, \cdots, y_j, m)$ we can derive $P_{\sigma,j}(0, 0, \cdots, 0, y_r, y_{r+1}, \cdots, y_j, m)$ for any assigned m. Accordingly if we can derive $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m)$ from $P_{\sigma,j}(0, 0, \cdots, 0, y_r, y_{r+1}, \cdots, y_j, n)$ then that formula can also be derived from $P_{\sigma,j}(0, 0, \cdots, 0, 0, y_{r+1}, \cdots, y_j, n)$. But we have seen that $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m)$ can be derived from $P_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n)$, and therefore $P_{\sigma,j}(x, y_0, y_1, \cdots, y_j, m)$ can be derived from the proved proposition $P_{\sigma,j}(0, 0, \cdots, 0, n)$. This derivation is completely finitist, and in fact it can readily be seen that, starting with the proved proposition $P_{\sigma,j}(0, 0, \cdots, 0, n)$ and substituting repeatedly in this and in the formulae (g) and (h) definite numerals for the variables $x, y_0, y_1, \cdots, y_j, n$, we derive the formula $P_{\sigma,j}(\alpha, \beta_0, \beta_1, \cdots, \beta_j, \mu)$, for assigned numerals $j, \alpha, \beta_0, \beta_1, \cdots, \beta_j, \mu$ and an assigned $\sigma(n)$, after exactly $N_{\sigma,j}(\alpha, \beta_0, \beta_1, \cdots, \beta_j, \mu)$ applications of the formulae

$P_{\sigma,j}(0, 0, \cdots, 0, n)$, (g), and (h), where $N_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n)$ is defined by the recursive equations:

$$N_{\sigma,j}(0, 0, \cdots, 0, n) = 1,$$

$$N_{\sigma,j}(x + 1, y_0, y_1, \cdots, y_j, n)$$
$$= N_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n + f_{\sigma,j}(0, y_0, y_1, \cdots, y_j, n)) + 1,$$

$$N_{\sigma,j}(0, 0, 0, \cdots, 0, y_r + 1, y_{r+1}, \cdots, y_j, n)$$
$$= N_{\sigma,j}(R \dot{-} 1, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j,$$
$$n + f_{\sigma,j}(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n)) + 1.$$

Thus the restricted ordinal theorem is proved for ordinals less than ω^{ω^ω}. Since

$$\{\sigma(n)\}^{[\sigma(n)]^{\sigma(n)}} = R\{\sigma(n)\}^{R+R\sigma(n)+R[\sigma(n)]^2+\cdots+R[\sigma(n)]^R}$$
$$+ \{\sigma(n)\}^{R+R\sigma(n)+R[\sigma(n)]^2+\cdots+R[\sigma(n)]^R},$$

where $R = \sigma(n) \dot{-} 1$, therefore the sequence $\zeta_\sigma(n, r)$, with

$$\zeta_\sigma(n, 0) = \{\sigma(n)\}^{[\sigma(n)]^{\sigma(n)}},$$
$$\zeta_\sigma(n, r + 1) = S^{\sigma(n+r)}_{\sigma(n+r+1)}[\zeta_\sigma(n, r) \dot{-} 1]$$

reaches zero in

$$f_{\sigma,R}(0, R, R, \cdots, R, n) + f_{\sigma,R}(R \dot{-} 1, R, R, \cdots, R,$$
$$n + f_{\sigma,R}(0, R, R, \cdots, R, n)) = f_{\sigma,R}(0, 0, \cdots, 0, R + 1, n)$$

steps, which completes the proof for ordinals less than or equal to ω^{ω^ω}.

We shall now show that the formula for the number of terms in the sequence commencing with $\{\sigma(n)\}^{[\sigma(n)]^{\sigma(n)}}$ is the same as the formula $f_{\sigma,j}(0, 0, \cdots, 0, 1, n)$ for the sequence commencing with $\{\sigma(n)\}^{[\sigma(n)]^j}$, $j < \sigma(n)$, with j replaced by $\sigma(n)$; i.e., that

$$f_{\sigma,R}(0, 0, \cdots, 0, R + 1, n) = f_{\sigma,\sigma(n)}(0, 0, \cdots, 0, 1, n).$$

First we prove the identity:

$$f_{\sigma,j}(x, y_0, y_1, \cdots, y_j, n) = f_{\sigma,j+1}(x, y_0, y_1, \cdots, y_j, 0, n).$$

Let $E_\sigma(x, y_0, y_1, \cdots, y_j, n)$ assert this identity. By equation (iv), $E_\sigma(0, 0, \cdots, 0, n)$ holds, and by equation (v) and induction we derive $E_\sigma(x, y_0, y_1, \cdots, y_j, n)$ from $E_\sigma(0, y_0, y_1, \cdots, y_j, n)$. Furthermore by equation (vi) we derive $E_\sigma(0, 0, \cdots, 0, y_r + 1, y_{r+1}, \cdots, y_j, n)$ from $E_\sigma(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n)$ and $E_\sigma(R \dot{-} 1, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n + f_{\sigma,j}(0, R, R, \cdots, R, y_r, y_{r+1}, \cdots, y_j, n))$, and so the proof of $E_\sigma(x, y_0, y_1, \cdots, y_j, n)$ follows exactly as the proof of

$P_{\epsilon,j}(x, y_0, y_1, \cdots, y_j, n)$ above. Hence

$$\begin{aligned} f_{\epsilon,j+1}(0, 0, \cdots, 0, 1, n) = f_{\epsilon,j+1}(0, R, R, \cdots, R, 0, n) \\ + f_{\epsilon,j+1}(R \dot{-} 1, R, R, \cdots, R, 0, n + f_{\epsilon,j+1}(0, R, R, \cdots, R, 0, n)) \\ = f_{\epsilon,j}(0, R, R, \cdots, R, n) + f_{\epsilon,j}(R \dot{-} 1, R, R, \cdots, R, \\ n + f_{\epsilon,j}(0, R, R, \cdots, n)) \\ = f_{\epsilon,j}(0, 0, \cdots, 0, R + 1, n), \end{aligned}$$

and from this the required result follows by taking $j = R$.

The method of proof readily extends to ordinals beyond ω^{ω^ω}, but to reach

$$\omega^{\omega^{\omega^\omega}}$$

by these means seems hardly to be worth the labour involved. On the other hand it seems likely that a more subtle approach would enable the theorems to be proved, by finitist methods, for ordinals up to any assigned v_n, where $v_0 = \omega$, $v_{n+1} = \omega^{v_n}$. The important point revealed by the foregoing proofs is that if a function $g(k, n)$ specifies the number of terms in a decreasing sequence commencing with some $F(k, \sigma(n))$, $k < \sigma(n)$, then $F(\sigma(n), \sigma(n))$ is followed by a decreasing sequence of at most $g(\sigma(n), n)$ terms, so that from a proof of the restricted ordinal theorem for ordinals less than or equal to $\Omega(k)$ we derive a proof of the theorem for ordinals less than or equal to $\Omega(\omega)$. The position appears to be, therefore, that if $P(n)$ expresses the restricted ordinal theorem for ordinals less than or equal to v_n, then $P(n)$ is capable of a finite constructive proof for *any assigned* n, but $(n)P(n)$ is not so provable—which of course involves that in the "reine Zahlentheorie," there can be no general formula $G(k, n)$ with a free variable k, specifying the number of terms in a decreasing sequence commencing with the ordinal v_k, but only specific formulae for particular values of k.

The formula $P_\epsilon(x + 1, p, n)$ above can be derived from the formulae (a), (b), (c) in a purely formal manner by means of recursive number theory. The following derivation was communicated to me by Professor Bernays.

Let $f(x, p, n)$, $g(p, n)$, $h(n)$ be recursive functions, and let $P(x, p, n)$ be an abbreviation for the equation $f(x, p, n) = 0$. Then $P(x, p, n)$ will be derived from the formulae:

$$P(0, 0, n), \tag{a*}$$

$$P(0, p, n) \ \& \ P(x, p, g(p, n)) \rightarrow P(x + 1, p, n) \tag{b*}$$

$$P(0, p, n) \ \& \ P(h(n), p, g(p, n)) \rightarrow P(0, p + 1, n). \tag{c*}$$

Define $\psi(x, p, n)$ by the equations,

$$\psi(0, p, n) = n,$$

$$\psi(x + 1, p, n) = \psi(x, p, g(p, n)),$$

and let $Q(x, p, n)$ be an abbreviation for

$$\sum_{z \leq x} f(0, p, \psi(z, p, n)) = 0.$$

From the demonstrable formula,

$$\sum_{z \leq x+1} f(0, p, \psi(z, p, n)) = f(0, p, n) + \sum_{z \leq x} f(0, p, \psi(z + 1, p, n)),$$

and the definitions of $\psi(x, p, n)$ and $Q(x, p, n)$ we derive

$$Q(x + 1, p, n) \rightarrow P(0, p, n) \,\&\, Q(x, p, g(p, n)), \tag{d*}$$

which together with (b*) gives

$$\{Q(x, p, g(p, n)) \rightarrow P(x, p, g(p, n))\} \rightarrow \{Q(x + 1, p, n) \rightarrow P(x + 1, p, n)\}$$

and this formula, in conjunction with the demonstrable formula $Q(0, p, n) \rightarrow P(0, p, n)$, gives $Q(x, p, n) \rightarrow P(x, p, n)$, by means of that schema of generalised induction referred to above. (The application of Skolem's schema II is not quite immediate, since the two parameters p, n must first be reduced to a single parameter by the method explained by Hilbert and Bernays for the case of primitive recursion.[7]) From $Q(x, p, n) \rightarrow P(x, p, n)$ and the formulae (c*) and (d*) we derive:

$$Q(h(n) + 1, p, n) \rightarrow P(0, p + 1, n). \tag{e*}$$

From (e*) and (a*) we derive $P(0, p, n)$ by the schema of generalised induction briefly discussed, at the end of the paper of Skolem's to which we have already referred,[8] as being reducible to schema II and ordinary induction. To carry out this reduction we make the following definitions. $K(p, n)$ is an abbreviation for

$$\sum_{u \leq n} \sum_{z \leq h(u)+1} f(0, p, \psi(z, p, u)) = 0,$$

and

$$\mu(p, n) = \operatorname*{Max}_{u \leq n} \operatorname*{Max}_{z \leq h(u)+1} \psi(z, p, u).$$

From the definitions of $K(p, n)$ and $\psi(x, p, n)$ there follows

$$K(p, n) \rightarrow P(0, p, n), \tag{k$_0$}$$

and using (a*) and the definition of $Q(x, p, n)$ we obtain

$$K(0, n). \tag{k$_1$}$$

Furthermore the definitions of $K(p, n)$, $Q(x, p, n)$ and formula (e*) yield

$$K(p, k) \rightarrow (\sum_{x \leq k} f(0, p + 1, x) = 0),$$

[7] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, vol. I, Berlin 1934, p. 322.
[8] In footnote 5. Vide pp. 275–276.

whence

$$K(p, \mu(p+1, n)) \rightarrow \{ \sum_{u \leq n} \sum_{z \leq h(u)+1} f(0, p+1, \psi(z, p+1, u)) = 0 \},$$

i.e.,

$$K(p, \mu(p+1, n)) \rightarrow K(p+1, n). \qquad (k_2)$$

Formulae (k_1) and (k_2) give $K(p, n)$, by the generalised induction schema II, whence by (k_0) we derive $P(0, p, n)$, i.e., $f(0, p, n) = 0$, whence $Q(x, p, n)$. Finally $P(x, p, n)$ is derived from the proved formulae $Q(x, p, n)$, $Q(x, p, n) \rightarrow P(x, p, n)$. Taking

$$(\sigma(n) \dot{-} (x+1)) \cdot (\sigma(n) \dot{-} p)) \cdot \gamma_\sigma(x+1, p, n, f_\sigma(x+1, p, n))$$

for $f(x, p, n)$, $n + f_\sigma(1, p, n)$ for $g(p, n)$, and $\sigma(n) \dot{-} 1$ for $h(n)$, it follows that

$$(\sigma(n) \dot{-} (x+1)) \cdot (\sigma(n) \dot{-} p)) \cdot \gamma_\sigma(x+1, p, n, f_\sigma(x+1, p, n)) = 0$$

is proved, and this equation is a formal transform of the formula $P_\sigma(x+1, p, n)$.

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THE UNIVERSITY, READING, ENGLAND