Complex power series: an example

The complex logarithm

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Purpose of this lecture

- Recall notions about convergence of real sequences and series
- Introduce these notions for complex sequences and series
- Illustrate these using the Taylor series of $\mathrm{Log}(1+z)$

A readable version of these slides can be found via

`http://fa.its.tudelft.nl/~hart`

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The definition

Definition

The sequence $\{x_n\}$ converges to the real number x, in symbols,

$$\lim_{n\to\infty} x_n = x$$

means: for every positive ϵ there is a natural number N such that for all $n \geqslant N$ one has $|x_n - x| < \epsilon$.

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Well-known examples

The following should be well-known:

- $\lim_{n\to\infty} \frac{1}{n} = 0$
- more generally $\lim_{n\to\infty} \frac{1}{n^p} = 0$ when $p > 0$.
- $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$

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A useful example

We show $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Note $\sqrt[n]{n} > 1$, so $a_n = \sqrt[n]{n} - 1$ is positive.

Apply the binomial formula:

$$n = (1 + a_n)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k = 1 + na_n + \frac{1}{2}n(n-1)a_n^2 + \cdots$$

we drop all terms but the second ...

A useful example

… and we find $n \geqslant \frac{1}{2}n(n-1)a_n^2$, and hence $a_n^2 \leqslant \frac{2}{n-1}$.

Take square roots: $0 < a_n < \frac{\sqrt{2}}{\sqrt{n-1}}$.

By the Squeeze Law: $\lim_{n\to\infty} a_n = 0$.

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Definition

The definition is identical (modulus replaces absolute value).

Definition

The sequence $\{z_n\}$ converges to the complex number z, in symbols,

$$\lim_{n\to\infty} z_n = z$$

means: for every positive ϵ there is a natural number N such that for all $n \geqslant N$ one has $|z_n - z| < \epsilon$.

Example: z^n

If $z \in \mathbb{C}$ then

- $\lim_{n\to\infty} z^n = 0$ if $|z| < 1$
- $\lim_{n\to\infty} z^n$ does not exist if $|z| = 1$ and $z \neq 1$
- $\lim_{n\to\infty} z^n = \infty$ if $|z| > 1$

$\lim_{n\to\infty} z_n = \infty$ means: for every positive M there is a natural number N such that for all $n \geqslant N$ one has $|z_n| > M$.

Oh yes: $\lim_{n\to\infty} 1^n = 1$.

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Series

Given a sequence $\{z_n\}$ what does (or should)

$$z_0 + z_1 + z_2 + z_3 + \cdots$$

mean?

Make a new sequence $\{s_n\}$ of *partial sums*:

$$s_n = \sum_{k \leqslant n} z_k$$

Convergence

If $\sigma = \lim_n s_n$ exists then we say that the *series* $\sum z_n$ converges and we write $\sigma = \sum_n z_n$.

Thus we give a meaning to $z_0 + z_1 + z_2 + z_3 + \cdots$:

the limit (if it exists) of the sequence of partial sums.

This definition works for real and complex sequences alike.

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Geometric series

Fix z and consider $1 + z + z^2 + z^3 + \cdots$ (so $z_n = z^n$).

The partial sums can be calculated explicitly:

$$s_n = 1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

for $z = 1$ we have $s_n = n + 1$.

Geometric series

The limit of the sequence of partial sums is easily found, in most cases:

- $|z| < 1$: $\sum_n z^n = \frac{1}{1-z}$
- $|z| > 1$: $\sum_n z^n = \infty$ also if $z = 1$

Geometric series

if $|z| = 1$ and $z \neq 1$ then the limit does not exist but we do have

$$|s_n| \leqslant \frac{2}{|1 - z|}$$

so for each individual z the partial sums are bounded

the bound is also valid if $|z| < 1$.

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Further examples

- $\sum_n \frac{1}{n} = \infty$ (even though $\lim_n \frac{1}{n} = 0$)
- $\sum_n \frac{(-1)^{n+1}}{n} = \ln 2$ (as we shall see later)
- $\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler)
- $\sum_n \frac{1}{n!} = e$
- $\sum_n \frac{1}{n^p}$ converges iff $p > 1$

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Absolute convergence

Absolute convergence: $\sum_n |z_n|$ converges.

Absolute convergence implies convergence
(but not necessarily conversely).

$\sum_n \frac{(-1)^n}{n}$ converges but $\sum_n \frac{1}{n}$ does not

We shall see: $\sum_n \frac{z^n}{n}$ converges for all z with $|z| = 1$ and $z \neq 1$.

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Comparison test

comparison if $|z_n| \leqslant a_n$ for all n
and $\sum_n a_n$ converges
then $\sum_n z_n$ converges absolutely

Pointwise convergence

A sequence $\{f_n\}$ of functions converges to a function f (on some domain) if for each individual z in the domain one has

$$\lim_{n\to\infty} f_n(z) = f(z)$$

Standard example: $f_n(z) = z^n$ on $D = \{z : |z| < 1\}$.
We know $\lim_n f_n(z) = 0$ for all $z \in D$, so $\{f_n\}$ converges to the zero function.

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Uniform convergence

$f_n(z) \to f(z)$ *uniformly* if for every $\epsilon > 0$ there is an $N(\epsilon)$ such that for all $n \geqslant N(\epsilon)$ we have

$$\left|f_n(z) - f(z)\right| < \epsilon$$

for all z in the domain.

Important fact: if $f_n \to f$ uniformly and each f_n is continuous then so is f.

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Uniform convergence: standard example

We have $z^n \to 0$ on D but not uniformly: let $\epsilon = \frac{1}{2}$,
for every n let $z_n = \sqrt[n]{\frac{1}{2}}$ then $|f_n(z_n) - f(z_n)| = \frac{1}{2}$.

Let $r < 1$ and consider $D_r = \{z : |z| < r\}$;
then $z^n \to 0$ *uniformly* on D_r.

Given $\epsilon > 0$, take N such that $r^N < \epsilon$, then for $n \geqslant N$ and all $z \in D_r$ we have

$$|z^n| \leqslant r^n \leqslant r^N < \epsilon$$

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Uniform convergence: standard example

$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for each $z \in D$ but not uniformly:

$$\sum_{k=0}^{n} z^k - \frac{1}{1-z} = \sum_{k=n+1}^{\infty} z^k = \frac{z^{n+1}}{1-z}$$

For each individual n this difference is unbounded.

Uniform convergence: standard example

On a smaller disk D_r we have

$$\left|\sum_{k=0}^{n} z^k - \frac{1}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right| \leqslant \frac{r^{n+1}}{1-r}$$

So, on D_r the series does converge *uniformly*.

Uniform convergence: M-test

Very useful test: if there is a convergent series $\sum_n M_n$ such that

$$|f_n(z)| \leqslant M_n$$

for all z in the domain, then $\sum_n f_n$ converges absolutely and uniformly on the domain.

Previous example: $|z^n| \leqslant r^n$ for all $z \in D_r$.

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Power series

Special form: a fixed number z_0 and a sequence $\{a_n\}$ of numbers are given. Put $f_n(z) = a_n(z - z_0)^n$, we write

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for the resulting series.

Radius of convergence

Important fact:
if $\lim_n a_n(w - z_0)^n = 0$ for some w then $\sum_n a_n(z - z_0)$ converges *absolutely* whenever $|z - z_0| < |w - z_0|$.

Use comparison test:
first fix N such that $|a_n(w - z_0)^n| \leqslant 1$ for $n \geqslant N$.
Then

$$\left|a_n(z - z_0)^n\right| = \left|a_n(w - z_0)^n \left(\frac{z - z_0}{w - z_0}\right)^n\right| \leqslant \left|\frac{z - z_0}{w - z_0}\right|^n$$

Geometric series with $r < 1$.

Radius of convergence

Even better: if $r < |w - z_0|$ then the power series converges *uniformly* on the disc

$$D(z_0, r) = \{z : |z - z_0| \leqslant r\}$$

Same proof gives

$$\left|a_n(z - z_0)^n\right| \leqslant \left(\frac{r}{|w - z_0|}\right)^n$$

for all z in the disc, apply the M-test.

Radius of convergence

Theorem

Given a power series $\sum_n a_n(z-z_0)^n$ *there is an* R *such that*

- $\sum_n a_n(z-z_0)^n$ *converges if* $|z-z_0| < R$
- $\sum_n a_n(z-z_0)^n$ *diverges if* $|z-z_0| > R$

In addition: if $r < R$ *then the series converges uniformly on* $\{z : |z-z_0| \leqslant r\}$.

- On the boundary — $|z-z_0| = R$ — anything can happen.
- $R = 0$, $0 < R < \infty$ and $R = \infty$ are all possible.

R is the *radius of convergence* of the series.

Examples

- $\sum_n z^n$: $R = 1$
- $\sum_n \frac{1}{n} z^n$: $R = 1$
- $\sum_n n z^n$: $R = 1$
- $\sum_n \frac{1}{n!} z^n$: $R = \infty$
- $\sum_n n^n z^n$: $R = 0$

In each case consider $\lim_n a_n z^n$ for various z.

On the boundary

The series $\sum_n z^n$ and $\sum_n nz^n$ both have radius 1.

What happens when $|z| = 1$?

The series diverges for all such z
as neither $\lim_n z^n$ nor $\lim_n nz^n$ is ever zero.

On the boundary

The series $\sum_n \frac{1}{n} z^n$ has radius 1.

What happens when $|z| = 1$?

That depends on z.

For $z = 1$ we have divergence: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Remember: $\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{1}{2}k$ for all k

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On the boundary

The series $\sum_n \frac{1}{n} z^n$ has radius 1.

What happens when $|z| = 1$ and $z \neq 1$?

The series converges.

This will require some work.

Summation by parts

Remember integration by parts: $\int fg = Fg - \int Fg'$.

The same can be done for sums: let $\{a_n\}$ and $\{b_n\}$ be two sequences. We find a similar formula for $\sum_{n=k}^{l} a_n b_n$.

The integral is replaced by the sequence of partial sums: $A_n = \sum_{k=0}^{n} a_k$ (and $A_{-1} = 0$).

The derivative is replaced by the sequence of differences: $\{b_{n+1} - b_n\}$

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Summation by parts

The formula becomes

$$\sum_{n=k}^{l} a_n b_n = A_l b_l - A_{k-1} b_k - \sum_{n=k}^{l-1} A_n (b_{n+1} - b_n)$$

The proof consists of some straightforward manipulation.

We use this with $a_n = z^n$ and $b_n = \frac{1}{n}$, so $A_n = \frac{1-z^{n+1}}{1-z}$

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Back to the boundary

Fix some k and let $l > k$ be arbitrary.

$$\left|\sum_{n=k}^{l} \frac{1}{n} z^n\right| = \left|A_l \frac{1}{l} - A_{k-1}\frac{1}{k} - \sum_{n=k}^{l-1} A_n \Bigl(\frac{1}{n+1} - \frac{1}{n}\Bigr)\right|$$

$$\leqslant \frac{2}{|1-z|}\left(\frac{1}{l} + \frac{1}{k} + \sum_{n=k}^{l-1}\Bigl(\frac{1}{n} - \frac{1}{n+1}\Bigr)\right)$$

$$= \frac{2}{|1-z|}\frac{2}{k}$$

This holds for all l, so ...

Back to the boundary

... the partial sums $\sum_{n=1}^{k} \frac{1}{n} z^n$ form a Cauchy-sequence.

The *completeness* of the complex plane ensures that

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{n} z^n$$

exists. We denote the sum, for now, by $\sigma(z)$.

The series converges for all z with $|z| \leqslant 1$ and $z \neq 1$.

Uniform convergence

The inequality

$$\left|\sum_{n=k}^{l}\frac{1}{n}z^n\right| \leqslant \frac{4}{k|1-z|}$$

holds for every z with $|z| \leqslant 1$ and $z \neq 1$.

This implies uniform convergence on sets of the form

$$E_r = \{z : |z| \leqslant 1, |1-z| \geqslant r\}$$

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Uniform convergence

For $z \in E_r$ we have

$$\left|\sigma(z) - \sum_{n=1}^{k} \frac{1}{n} z^n\right| = \left|\sum_{n=k+1}^{\infty} \frac{1}{n} z^n\right| \leqslant \frac{4}{(k+1)|1-z|} \leqslant \frac{4}{(k+1)r}$$

Now, given $\epsilon > 0$ we take n so large that $\frac{4}{(N+1)r} < \epsilon$.

Then $\left|\sigma(z) - \sum_{n=1}^{k} \frac{1}{n} z^n\right| < \epsilon$ whenever $k \geqslant N$ and $z \in E_r$.

Integrating the geometric series

We know

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad (|z| < 1)$$

We also know

$$\frac{1}{1-z} = \sum_{n=0}^{k} z^n + \frac{z^{k+1}}{1-z}$$

Integrate this along the straight line L from 0 to z:

$$-\operatorname{Log}(1-z) = \sum_{n=0}^{k} \frac{1}{n+1} z^{n+1} + \int_L \frac{w^{k+1}}{1-w} \,\mathrm{d}w$$

Integrating the geometric series

We can find an (easy) upper bound for the absolute value of the integral:

$$\left|\int_L \frac{w^{k+1}}{1-w}\,\mathrm{d}w\right| \leqslant |z| \times \frac{|z|^{k+1}}{1-|z|} = \frac{|z|^{k+2}}{1-|z|}$$

Thus

$$\lim_{k\to\infty} \int_L \frac{w^{k+1}}{1-w}\,\mathrm{d}w = 0$$

and so . . .

Integrating the geometric series

... we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n = -\operatorname{Log}(1-z) \qquad (|z|<1)$$

but, by continuity of the sum function $\sigma(z)$ and the Logarithm this formula holds when $|z| = 1$ and $z \neq 1$ as well.

Often z is replaced by $-z$ and an extra minus sign is added to give

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n = \operatorname{Log}(1+z) \qquad (|z| \leqslant 1, z \neq -1)$$

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ln 2

As promised:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\operatorname{Log} 2 = -\ln 2$$

(use $z = -1$) or, with an extra minus sign:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

this is also written

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

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Rest of the boundary

If $z = e^{i\theta}$, with $\theta \neq 2k\pi$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{in\theta} = -\operatorname{Log}(1 - e^{i\theta}) = -\ln|1 - e^{i\theta}| - i\operatorname{Arg}(1 - e^{i\theta})$$

If we split the series and its sum into their respective real and imaginary parts we get two nice formulas.

Real part

Note

$$|1 - e^{i\theta}|^2 = (1 - e^{i\theta})(1 - e^{-i\theta}) = 2 - 2\cos\theta = 4\sin^2\frac{1}{2}\theta$$

So that $-\ln|1 - e^{i\theta}| = -\ln\bigl(2|\sin\frac{1}{2}\theta|\bigr)$ and we get

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n} = -\ln\Bigl(2|\sin\frac{1}{2}\theta|\Bigr)$$

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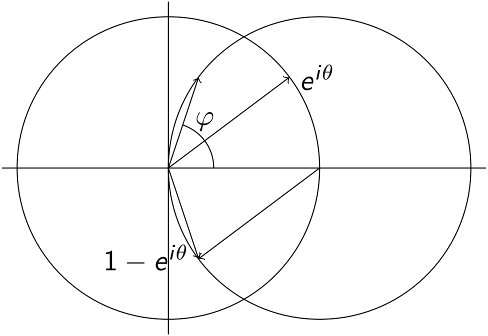
Imaginary part

To see what $\varphi = -\operatorname{Arg}(1 - e^{i\theta})$ is draw a picture



Imaginary part

We have $1 - e^{i\theta} = (1 - \cos\theta) - i\sin\theta$, so that if $0 < \theta < \pi$ we get

$$\tan\varphi = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \tan(\frac{1}{2}\pi - \frac{1}{2}\theta)$$

If $0 < \theta < \pi$ then φ and $\frac{1}{2}\pi - \frac{1}{2}\theta$ lie between 0 and $\frac{1}{2}\pi$ so that

$$\varphi = \frac{1}{2}\pi - \frac{1}{2}\theta$$

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Imaginary part

We find

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{1}{2}\pi - \frac{1}{2}\theta \qquad (0 < \theta < \pi)$$

The sum must be an odd function so

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = -\frac{1}{2}\pi - \frac{1}{2}\theta \qquad (-\pi < \theta < 0)$$