Delft University of Technology
Faculty EEMCS
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Exam Complex Analysis (wi4243AP/wi4244AP)
Wednesday 22 January 2014; 14:00 – 17:00.

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This exam consists of six questions.
Marks per question: see margin.
Resources allowed: calculator

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1. Let α be a complex number such that $|\alpha| \neq 1$ and consider the bilinear transformation given by

$$w = \frac{\alpha z - 1}{z - \overline{\alpha}}.$$

(2) a. Show that this transformation maps the unit circle onto itself.

(2) b. What is the image of the unit disk under this transformation?

(2) c. How does w traverse the unit circle as z traverses the unit circle in the positive direction?

(3) d. Now let $\alpha = \frac{1}{2}i$. Determine the image of the part of the unit disk that lies in the first quadrant under the transformation.

2. Define $u(x, y) = e^x(x \cos y - y \sin y)$

(2) a. Verify that u is harmonic.

(5) b. Determine all analytic functions that have u as their real part and write these as functions of z.

(5) 3. a. Let h be an analytic map from the unit disc $D = \{z : |z| \leqslant 1\}$ to the disc $E = \{z : |z| \leqslant 2\}$ that $h(0) = 0$. Show that $\bigl|h(z)\bigr| \leqslant |2z|$ for $z \in D$ and $\bigl|h'(0)\bigr| \leqslant 2$. *Hint*: Consider the function $h(z)/z$.

(5) b. Let f be an analytic map from the unit disc $D = \{z : |z| \leqslant 1\}$ to the disc $E = \{z : |z| \leqslant 2\}$ and let α be such that $|\alpha| < 1$ and $f(\alpha) = 0$. Show that $\bigl|f'(\alpha)\bigr| \leqslant \frac{2}{1-|\alpha|^2}$. *Hint*: Consider the function $g(z) = f\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right)$.

(8) 4. Evaluate the following integral

$$\int_0^{2\pi} \frac{1}{1 + 8\cos^2\theta}\, d\theta$$

Give all details.

(8) 5. Let θ be a real number in the interval $(0, \pi)$. Calculate the following Fourier transform

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 - 2x\cos\theta + 1}\, dx.$$

Deal with the case $\omega > 0$ only and give all details.

6. We consider the many-valued function $w = \sqrt{z^2 - 1}$.

(3) a. Suppose we use the branch of $\sqrt{\ }$ that has the positive real axis as its branch cut and is such that $\sqrt{-1} = i$. Determine the image of the open upper half plane, $\{z : \operatorname{Im} z > 0\}$, under this mapping.

This problem continues on the next page.

From now on we use the principal branch of $\sqrt{\ }$, that is, the negative real axis is the branch cut and $\sqrt{1} = 1$.

(3) b. Show that

$$f(z) = (z+1)\sqrt{\frac{z-1}{z+1}}$$

defines a branch of our function w with branch cut $[-1,1]$. What is the value of $f(2)$?

(3) c. Determine the first four terms of the Laurent series of this branch in the annulus $\{z : |z| > 1\}$.
Hint: $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$ if x is real and $|x| < 1$.

(3) d. Calculate

$$\oint_S f(z)\,dz$$

where S is the square with vertices at $\pm 5 \pm 5i$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 6}{6}$$

and rounded in the standard way.

THE END