ULTRAFILTERS AND COMBINATORICS

ULTRAFILTERS

X A SET (TODAY: $\mathbb{N}$ OR $\mathbb{N} \times \mathbb{N}$)

AN ULTRAFILTER ON X IS A FAMILY $\mathcal{U}$ OF SUBSETS SATISFYING

$$\left.\begin{array}{l} \bullet\ \mathcal{U} \neq \emptyset \text{ AND } \emptyset \notin \mathcal{U} \\ \bullet\ A, B \in \mathcal{U} \rightarrow A \cap B \in \mathcal{U} \\ \bullet\ A \in \mathcal{U},\ B \supseteq A \rightarrow B \in \mathcal{U} \end{array}\right\} \text{FILTER}$$

- ONE (HENCE ALL) OF THE FOLLOWING (EQUIVALENT) CONDITIONS
  - $(\forall A)(A \in \mathcal{U} \vee X \setminus A \in \mathcal{U})$
  - $(\forall A, B)((A \cup B \in \mathcal{U}) \rightarrow (A \in \mathcal{U} \vee B \in \mathcal{U}))$
  - $\mathcal{U}$ IS MAXIMAL IN THE CONGERIES OF FILTERS ON X

ULTRAFILTER THEOREM

EVERY FILTER CAN BE EXTENDED TO AN ULTRAFILTER

PROOF: ZORN'S LEMMA

TRIVIAL ULTRAFILTERS

$$\mathcal{U}_x = \{ A \subseteq X : x \in X \}$$

NON-TRIVIAL ULTRAFILTERS

EXTEND $\mathrm{Cof} = \{ A \subseteq X : |X \setminus A| < \omega \}$

TO AN ULTRAFILTER

NON-TRIVIAL ULTRAFILTERS ARE

HARD TO DESCRIBE:

DEFINE $\quad r : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$

BY $\quad r(A) = \sum_{n \in A} 2^{-n}$

SIERPIŃSKI (1938)

IF $\mathcal{U}$ IS A NON-TRIVIAL ULTRAFILTER

THEN $\{ r(A) : A \in \mathcal{U} \}$ IS NOT

LEBESGUE-MEASURABLE.

INTUITION: MEMBERS OF AN ULTRAFILTER

ARE 'LARGE'.

$\mu : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$ DEFINED BY

$\mu(A) = 1$ IF $A \in \mathcal{U}$, $\mu(A) = 0$ IF $A \notin \mathcal{U}$

IS A FINITELY ADDITIVE MEASURE.

Ramsey's Theorem

If $[\mathbb{N}]^2 = A_0 \cup A_1$ then there is an infinite $H \subseteq \mathbb{N}$ such that $[I]^2 \subseteq A_0$ or $[I]^2 \subseteq A_1$.

Proof.

Fix a non-trivial ultrafilter $\mathcal{U}$.

For $n \in \mathbb{N}$ let $\varepsilon_n \in \{0,1\}$ be such that $U_n = \{m > n : \{n, m\} \in A_{\varepsilon_n}\} \in \mathcal{U}$

Next pick $\varepsilon \in \{0,1\}$ such that

$$V = \{n : \varepsilon_n = \varepsilon\} \in \mathcal{U}$$

Now, recursively, define $\{n_i : i \in \mathbb{N}\}$ as follows

$$n_0 = \min V$$

$$n_1 = \min V \cap U_{n_0}$$

$$n_2 = \min V \cap U_{n_0} \cap U_{n_1}$$

$$n_3 = \min V \cap U_{n_0} \cap U_{n_1} \cap U_{n_2}$$

$$\vdots$$

$$n_i = \min V \cap \bigcap_{j<i} U_{n_j}$$

$$\vdots$$

SUMMARIZING

IF $i < j$ THEN $n_j \in U_{n_i}$

AND HENCE $\{n_i, n_j\} \in A_\varepsilon$.

WE FIND

$$I = \{n_i : i \in \mathbb{N}\}$$

IS AS REQUIRED.

HINDMAN'S THEOREM.

IF $\mathbb{N} = A_0 \cup A_1 \cup \dots \cup A_n$ THEN THERE ARE i AND AN INFINITE SET B SUCH THAT $FS(B) \subseteq A_i$.

$FS(B) = \{\sum x : x \subseteq B$ FINITE NONEMPTY$\}$

COROLLARY [FOLKMAN'S THEOREM]

FOR EVERY n AND k THERE IS $M(n, k)$ SUCH THAT: IF $\{0, 1, \dots, M(n,k)\} = A_0 \cup A_1 \cup \dots \cup A_n$ THEN THERE IS B OF SIZE k WITH $FS(B) \subseteq A_i$ FOR SOME i.

ADDING ULTRAFILTERS:

LET p AND q BE ULTRAFILTERS ON $\mathbb{N}$

IF $n \in \mathbb{N}$ AND $A \subseteq \mathbb{N}$ THEN

$$A - n = \{ m : n + m \in A \}.$$

- $n + q = \{ A : A - n \in q \}$

  ($n+q$ IS q SHIFTED n TO THE RIGHT)

- $p + q = \{ A : \{ n : A \in n+q \} \in p \}$

ASSUME WE HAVE p WITH $p = p + p$.

SO, $A \in p$ IFF $A \in p + p$

IFF $\{ n : A - n \in p \} \in p$

FOR $A \in p$ WRITE $A^* = \{ n : A - n \in p \}$

TAKE $A \in p$

SET $A_0 = A$ AND $n_0 = \min(A_0 \cap A_0^*)$

SO $n_0 \in A$ AND $A - n_0 \in p$

SET $A_1 = A_0 \cap A_0^* \cap (A_0 - n_1)$ AND

$n_1 = \min(A_1 \cap A_1^*) \setminus \{0, \ldots, n_0\}$

SO $n_1 \in A$ AND $n_0 + n_1 \in A$

SET $A_2 = A_1 \cap A_1^* \cap (A_1 - n_1)$ AND

$$n_2 = \min(A_2 \cap A_2^*)\setminus\{0, \dots, n_1\}$$

SO $n_2 \in A_2$

$n_0 + n_2 \in A_0$, $n_1 + n_2 \in A_1$

$n_0 + n_1 + n_2 \in A_0$

SET $A_3 = A_2 \cap A_2^* \cap (A_2 - n_2)$ AND

$$n_3 = \min(A_3 \cap A_3^*)\setminus\{0, \dots, n_2\}$$

SO $n_3 \in A_3$

$n_0 + n_3 \in A_0$, $n_1 + n_3 \in A_1$, $n_2 + n_3 \in A_2$,

$n_0 + n_1 + n_3 \in A_0$, $n_1 + n_2 + n_3 \in A_1$,

$n_0 + n_1 + n_2 + n_3 \in A_0$.

$B = \{ n_0, n_1, n_2, n_3, \dots \}$

SATISFIES

$FS(B) \subseteq A$.

WE SEE: EVERY ELEMENT OF p CONTAINS SOME $FS(B)$.

TO PROVE HINDMAN'S THEOREM:

FIND NON-TRIVIAL ULTRAFILTER p SATISFYING $p = p + p$.

- $\mathbb{N}^*$ IS THE SET OF NON-TRIVIAL UF'S.
- FOR $A \subseteq \mathbb{N}$ PUT $A^* = \{p \in \mathbb{N}^* : A \in p\}$; THEN $\{A^* : A \subseteq \mathbb{N}\}$ IS A BASE FOR A COMPACT HAUSDORFF TOPOLOGY.
- $(p, q) \mapsto p + q$ IS ASSOCIATIVE AND CONTINUOUS IN THE FIRST VARIABLE.
- $\mathcal{A} = \{F : F$ IS CLOSED AND $F + F \subseteq F\}$.

  – $\mathbb{N}^* \in \mathcal{A}$

  – IF $\mathcal{A}'$ IS A CHAIN IN $\mathcal{A}$ THEN $\bigcap \mathcal{A}' \in \mathcal{A}$

  – ZORN: TAKE A MINIMAL ELEMENT M

  – M CONSISTS OF ONE POINT

  LET $x \in M$.

  - $M + x$ IS CLOSED; $M + x \subseteq M$;

    $(M + x) + (M + x) = (M + x + M) + x \subseteq M + x$

    SO $M + x = M$

- $N_x = \{ y \in M : y + x = x \}$

  N_x IS NONEMPTY

  N_x IS CLOSED (CONTINUITY)

  $N_x + N_x \subseteq N_x$: $(z + y) + x = z + (y + x)$

  $= z + x = x$

  SO $N_x = M$ AND HENCE $x + x = x$.

VAN DER WAERDEN'S THEOREM

IF $\mathbb{N} = A_0 \cup A_1 \cup \cdots \cup A_n$ THEN THERE IS AN i SUCH THAT A_i CONTAINS ARBITRARILY LONG ARITHMETIC PROGRESSIONS.

ORDER THE IDEMPOTENTS BY

$p \leq q$ IFF $p = p + q = q + p$

THEOREM

IF p IS A MINIMAL IDEMPOTENT IN $\mathbb{N}^*$ AND $A \in p$ THEN A CONTAINS ARBITRARILY LONG ARITHMETIC PROGRESSIONS.

SO, VAN DER WAERDEN'S THEOREM FOLLOWS ONCE WE FIND A MINIMAL IDEMPOTENT.

PROOF OF THE THEOREM

WRITE $N = \mathbb{N} \setminus \{0\}$

DEFINE $\varphi_n : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ BY

$$\varphi_n(a, d) = a + nd$$

NOTE $\{a, a+d, \ldots, a+nd\} \in A$

IFF $(a,d) \in \varphi_0^{\leftarrow}[A] \cap \varphi_1^{\leftarrow}[A] \cap \cdots \cap \varphi_n^{\leftarrow}[A]$

TO FIND A PROPER A.P. GET $d \geq 1$.

WE FIND A MINIMAL IDEMPOTENT e

IN $(\mathbb{N} \times \mathbb{N})^*$ SUCH THAT $(\forall n)(\varphi_n(e) = p)$

- $\mathbb{N} \times N \in e$ (BY MINIMALITY)
- FOR ALL n WE HAVE $\varphi_n^{\leftarrow}[A] \in e$

  AND SO FOR ALL n WE GET

  $$\mathbb{N} \times N \cap \bigcap_{i \leq n} \varphi_i^{\leftarrow}[A] \neq \emptyset$$

SO THERE.

EXISTENCE OF MINIMAL IDEMPOTENTS

ABSTRACT SETTING:

$(S, *)$ COMPACT SEMIGROUP SUCH THAT $\rho_x : y \longmapsto y * x$ IS ALWAYS CONTINUOUS.

- LEFT IDEAL L: FOR ALL x ONE HAS $x * L \subseteq L$.
- EVERY LEFT IDEAL CONTAINS A CLOSED LEFT IDEAL: $x \in L \to S * x \subseteq L$
- ZORN: MINIMAL LEFT IDEALS EXIST - AND THEY ARE CLOSED; HENCE THEY CONTAIN IDEMPOTENTS.
- IF L IS A MINIMAL LEFT IDEAL AND $p \in L$ IS AN IDEMPOTENT THEN p IS MINIMAL.

NOTE: $L = S * p$ AND IF $x \in L$ THEN $x * p = x$: FOR $x = y * p$ SO $x * p = y * p * p$

$$= y * p$$
$$= x$$

IF $q \le p$, I.E., $q = q * p = p * q$
THEN $q \in L$, SO $S * q = L$ AND,
AS ABOVE, $x * q = x$ FOR $x \in L$.
WE GET $q = q * p = p * q = p$.

1. FIX A MINIMAL IDEMPOTENT p IN $\mathbb{N}^*$
2. $M = \{ x \in (\mathbb{N} \times \mathbb{N})^* : (\forall n)(\varphi_n(x) = p)\}$ IS A COMPACT SEMIGROUP
   PICK A MINIMAL IDEMPOTENT e IN M.
3. e IS MINIMAL IN $(\mathbb{N} \times \mathbb{N})^*$

IF $f + e = e + f = f$ THEN

$$\varphi_n(f) + p = p + \varphi_n(f) = \varphi_n(f)$$

AND $\varphi_n(f) + \varphi_n(f) = \varphi_n(f)$, SO $\varphi_n(f) = p$.
IT FOLLOWS THAT $f \in M$ AND HENCE $f = e$.

4. $T = \{ x : \mathbb{N}\times\mathbb{N} \in x \}$ IS A TWO-SIDED IDEAL: IF $(n,m) \in \mathbb{N}\times\mathbb{N}$ THEN

$\mathbb{N}\times\mathbb{N} - (n,m) = \mathbb{N}\times\mathbb{N}$ AND, ALSO,

$\mathbb{N}\times\mathbb{N} - (n,0) = \mathbb{N}\times\mathbb{N}$.

SO $x+y,\ y+x \in T$ IF $x \in T$.

5. $(\mathbb{N}\times\mathbb{N})^* + e \cap T \neq \emptyset$:

IF $x \in T$ THEN $x+e$ BELONGS TO THE INTERSECTION

6. THE INTERSECTION IS A LEFT-IDEAL SO $(\mathbb{N}\times\mathbb{N})^* + e \subseteq T$ AND HENCE, FINALLY, $e = e+e \in T$.

LITERATURE

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