A connected F-space

Non impeditus ab ulla scientia

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Outline

1. The main result
2. Why?
   - d-independent sets and d-bases
   - What does our space do then?
3. The construction
   - Intuition
   - Starting point
   - Thin out S_u
   - Create X
4. Sources

A space and a function

- There is a compact Hausdorff space, X, that is *connected* and an F-*space*.

A space and a function

- There is a compact Hausdorff space, X, that is *connected* and an F*-space*.
- It supports a continuous real-valued function, f, that is not *essentially constant*.

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- Yet, for f we have: $\Omega_f = \bigcup_t \operatorname{int} f^{\leftarrow}(t)$ is *not* dense. (This is *not* essentially constant.)

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d-independent sets

D, a subset of $C(X)$, is *d-independent* if for every nonempty open set O the nonzero elements in $\{d \restriction O : d \in D\}$ are linearly independent.

d-bases

A d-independent set D is a *d-basis* if for every $g \in C(X)$ there is a disjoint family $\mathcal{O}$ of open sets, with dense union, such that for every O the restriction $g \restriction O$ is a linear combination of (finitely many members of) $\{d \restriction O : d \in D\}$.

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Maximally independent does not mean base

- The family $\{1\}$ is maximally d-independent.
  (For every continuous function $g : X \to \mathbb{R}$ and every t in the interior of the *interval* $g[X]$ the interior of $g^{\leftarrow}(t)$ is nonempty.)

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  (For every continuous function $g : X \to \mathbb{R}$ and every t in the interior of the *interval* $g[X]$ the interior of $g^{\leftarrow}(t)$ is nonempty.)
- Yet, the family $\{\imath\}$ is not a d-basis.
  (For f we have: $\Omega_f = \bigcup_t \operatorname{int} f^{\leftarrow}(t)$ is *not* dense.)

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No (easy) projection

Using a d-basis that contains $\mathbb{1}$ one can project $C(X)$ onto the subspace of essentially constant functions, in case X is extremally disconnected.

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Apparently even more difficult for F-spaces.

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A rough picture

Think of X as the following subspace of S:

$$\big([0,1] \times \{0\}\big) \cup \big(C \times [0,1]\big)$$

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This 'X' is not an F-space ...

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- and extend to $\beta p : \beta\mathbb{S} \to [0,1]$.

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A component of $\beta\mathbb{S}$ and a function

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but S_u and βp are not good enough ...

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- $Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$ and $q_\alpha = q_0 \restriction Y_\alpha$ if α is a limit

There is a first (limit) $\delta < \mathfrak{c}^+$ where $Y_\delta = Y_{\delta+1}$, meaning that $\operatorname{int}_\delta q_\delta^{\leftarrow}(t) = \emptyset$ for all t

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Sadly, Y_δ is not connected
However, take the *bottom line* of S_u:

$$B_u = S_u \cap \mathrm{cl}\bigl(\omega \times [0,1] \times \{0\}\bigr).$$

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X is connected and F
$\operatorname{int} f^{\leftarrow}(t) \subseteq B_u$ for all t

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Here are X and f

Finally then

- $X = B_u \cup Y_\delta$
- $f = \beta p \restriction X$

X is connected and F
$\operatorname{int} f^{\leftarrow}(t) \subseteq B_u$ for all t
All components of Y_δ meet the top line, so $\Omega_f \subseteq B_u$ is not dense

Light reading

Website: `fa.its.tudelft.nl/~hart`

Y. A. Abramovich and A. K. Kitover.
d-Independence and d-bases, Positivity, **7** (2003), 95–97.

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