There is no categorical metric continuum

Non impeditus ab ulla scientia

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Outline

1. The main result
   - The statement
   - What it means
2. The Proof
   - Main lemma
   - Finishing up
3. Sources

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Given a metric continuum X there is another metric continuum Y such that

- X and Y look the same
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- X and Y are not homeomorphic

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Elementary equivalence

We consider bases that are closed under finite unions and intersections.
These are lattices and ‘elementary equivalence’ is with respect to the lattice structure.
Two lattices are ‘elementarily equivalent’ if they satisfy the same first-order sentences.

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\bigl((x \sqcap y = \mathbf{0}) \to ((x \leqslant u) \wedge (y \leqslant v) \wedge (u \sqcap v = \mathbf{0}) \wedge (u \sqcup v = \mathbf{1}))\bigr)$$

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In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).
By *compactness*, if some base satisfies this sentence then the space is zero-dimensional.

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If some base satisfies this sentence then the space has no isolated points.

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Example: the Cantor set is categorical

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Thus: if X looks like C then X is homeomorphic to C.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.
Then X is zero-dimensional and without isolated points.
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Thus: if X looks like C then X is homeomorphic to C.

The Cantor set is categorical among compact metric spaces.

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What the main result says

Among metric continua there is no categorical space.

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No (in)finite list of first-order properties will characterize a single metric continuum.

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Chainability is *not* first-order.
(Hereditary indecomposability is.)

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An embedding lemma

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
There is an embedding of $\mathcal{C}$ into the ultrapower of $\mathcal{B}$ by u.

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Ultrapower $\mathcal{B}_u$: the power $\mathcal{B}^\omega$ modulo the equivalence relation "$\{n : B_n = C_n\} \in u$".

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Ultrapower $\mathcal{B}_u$: the power $\mathcal{B}^\omega$ modulo the equivalence relation "$\{n : B_n = C_n\} \in u$".
Ultrapower theorem: $\mathcal{B}$ and $\mathcal{B}_u$ are elementarily equivalent.

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Let Y be the Wallman space of $\mathcal{D}$.

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- $\mathcal{D}$ is a base for the closed sets of Y (by Wallman's theorem).
- $\mathcal{D}$ is elementarily equivalent to $\mathcal{B}_u$ and hence to $\mathcal{B}$.
- Y maps onto Z (because $\varphi[\mathcal{C}]$ is embedded into $\mathcal{D}$).

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Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and such that Y maps onto Z.
So: Y is not homeomorphic to X.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

T. Banakh, P. Bankston, B. Raines and W. Ruitenburg.
Chainability and Hemmingsen's theorem,
`http://www.mscs.mu.edu/~paulb/Paper/chainable.pdf`

K. P. Hart.
There is no categorical metric continuum, to appear.

Z. Waraszkiewicz.
Sur un problème de M. H. Hahn, *Fundamenta Mathematicae* **22** (1934) 180–205.

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