Why can't we do $\int e^{-x^2}\,dx$?

Non impeditus ab ulla scientia

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Outline

1. Integration in finite terms
2. Formalizing the question
   - Differential fields
   - Elementary extensions
   - The abstract formulation
3. Applications
   - Liouville's criterion
   - $\int e^{-z^2}\,dz$ at last
   - Further examples
4. Sources
5. Technicalities

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Elementary functions: e^x, $\sin x$, x, $\log x$, ...

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- exponentials

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We do not look at all functions that we get in this way and check that their derivatives are not e^{-x^2}.

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We do establish an *algebraic* condition for a function to have a primitive function that is expressible in terms of elementary functions, as described above.

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We do establish an *algebraic* condition for a function to have a primitive function that is expressible in terms of elementary functions, as described above.

We then show that e^{-x^2} does not satisfy this condition.

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- $D(ab) = D(a)b + aD(b)$

Main example(s)

The rational (meromorphic) functions on (some domain in) $\mathbb{C}$, with $D(f) = f'$ (of course).

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We write $a' = D(a)$ in any differential field.

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Easy properties

Exercises

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- $1' = 0$ (Hint: $1' = (1^2)'$)
- The 'constants', i.e., the $c \in F$ with $c' = 0$ form a subfield

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- 'logarithmic derivative':

$$\frac{(a^m b^n)'}{a^m b^n} = m\frac{a'}{a} + n\frac{b'}{b}$$

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Much of Calculus is actually Algebra ...

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G is an elementary extension of F is $G = F(t_1, t_2, \ldots, t_N)$, where each time $F_i(t_{i+1})$ is a simple elementary extension of $F_i = F(t_1, \ldots, t_i)$.

Outline

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Elementary integrals

We say that $a \in F$ has an *elementary integral* if there is an elementary extension G of F with an element t such that $t' = a$.

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We say that $a \in F$ has an *elementary integral* if there is an elementary extension G of F with an element t such that $t' = a$.
The Question: characterize (of give necessary conditions for) this.

A characterization

Theorem (Rosenlicht)

Let F be a differential field of characteristic zero and $a \in F$. If a has an elementary integral **in some extension with the same field of constants** *then there are $v \in F$, constants $c_1, \ldots, c_n \in F$ and elements $u_1, \ldots u_n \in F$ such that*

$$a = v' + c_1 \frac{u_1'}{u_1} + \cdots + c_n \frac{u_n'}{u_n}.$$

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$$a = v' + c_1 \frac{u_1'}{u_1} + \cdots + c_n \frac{u_n'}{u_n}.$$

The converse is also true: $a = (v + c_1 \log u_1 + \cdots + c_n \log u_n)'$.

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Consider $\frac{1}{1+x^2} \in \mathbb{R}(x)$

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Comment on the constants

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- an elementary integral is

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  using a *larger* field of constants: $\mathbb{C}$
- there are *no* v, u_i and c_i in $\mathbb{R}(x)$ as in Rosenlicht's theorem.

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F is a transcendental extension of $\mathbb{C}(z)$.
If fe^g has an elementary integral then in F we must have

$$ft = v' + c_1\frac{u_1'}{u_1} + \cdots + c_n\frac{u_n'}{u_n}$$

with $c_1, \ldots, c_n \in \mathbb{C}$ and $v, u_1, \ldots, u_n \in \mathbb{C}(z,t)$.

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The criterion

Using *algebraic* considerations one can then get the following criterion.

Theorem (Liouville)

The function fe^g has an elementary integral iff there is a rational function $q \in \mathbb{C}(z)$ such that

$$f = q' + qg'$$

the integral then is qe^g (of course).

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In this case $f(z) = 1$ and $g(z) = -z^2$.

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In this case $f(z) = 1$ and $g(z) = -z^2$.
Is there a q such that $1 = q'(z) - 2zq(z)$?
Assume q has a pole β and look at principal part of Laurent series

$$\sum_{i=1}^{m} \frac{\alpha_i}{(z-\beta)^i}$$

Its contribution to the right-hand side should be zero.

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$\int e^{-z^2}\,dz$

We get, at the pole β:

$$0 = \sum_{i=1}^{m} \left(-\frac{i\alpha_i}{(z-\beta)^{i+1}} - \frac{2z\alpha_i}{(z-\beta)^i} \right)$$

Successively: $\alpha_1 = 0, \ldots, \alpha_m = 0$.

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Successively: $\alpha_1=0,\ \ldots,\ \alpha_m=0$.
So, q is a polynomial, but $1=q'(z)-2zq(z)$ will give a mismatch of degrees.

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$\int \frac{e^z}{z}\,dz$

Here $f(z) = 1/z$ and $g(z) = z$, so we need $q(z)$ such that

$$\frac{1}{z} = q'(z) + q(z)$$

Again, via partial fractions: no such q exists.

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Again, via partial fractions: no such q exists.
$\int e^{e^z}\,dz = \int \frac{e^u}{u}\,du = \int \frac{1}{\ln v}\,dv$
(substitutions: $u = e^z$ and $u = \ln v$)

$\int \frac{\sin z}{z}\,dz$

In the complex case this is just $\int \frac{e^z - e^{-z}}{z}\,dz$.

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In the complex case this is just $\int \frac{e^z - e^{-z}}{z}\,dz$.
Let $t = e^z$ and work in $\mathbb{C}(z,t)$; adapt the proof of the main theorem to reduce this to $\frac{1}{z} = q'(z) + q(z)$ with $q \in \mathbb{C}(z)$, still impossible.

Light reading

These slides at: `fa.its.tudelft.nl/~hart`

J. Liouville.
Mémoire sur les transcendents elliptiques considérées comme functions de leur amplitudes, Journal d'École Royale Polytechnique (1834)

M. Rosenlicht,
Integration in finite terms, American Mathematical Monthly, **79** (1972), 963–972.

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A useful lemma, I

Lemma

Let F be a differential field, $F(t)$ a differential extension with the same constants, with t transcendental over F and such that $t' \in F$.

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Let F be a differential field, $F(t)$ a differential extension with the same constants, with t transcendental over F and such that $t' \in F$. Let $f(t) \in F[t]$ be a polynomial of positive degree.

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Let F be a differential field, $F(t)$ a differential extension with the same constants, with t transcendental over F and such that $t' \in F$.
Let $f(t) \in F[t]$ be a polynomial of positive degree.
Then $f(t)'$ is a polynomial in $F[t]$ of the same degree as $f(t)$ or one less, depending on whether the leading coefficient of $f(t)$ is not, or is, a constant.

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Lemma

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- *for nonzero $a \in F$ and nonzero $n \in \mathbb{Z}$ we have $(at^n)' = ht^n$ for some nonzero $h \in F$;*

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- *for nonzero $a \in F$ and nonzero $n \in \mathbb{Z}$ we have $(at^n)' = ht^n$ for some nonzero $h \in F$;*
- *if $f(t) \in F[t]$ then $f(t)'$ is of the same degree as $f(t)$ and $f(t)'$ is a multiple of $f(t)$ iff $f(t)$ is a monomial (at^n).*

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$\int \frac{\sin x}{x}\,dz$, I

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$\int \frac{\sin x}{x}\,dz$, I

Write $F = \mathbb{C}(z)$ and $t = e^z$.
If $\int \frac{\sin z}{z}\,dz$ were elementary then

$$\frac{t^2-1}{tz} = v' + c_1\frac{u_1'}{u_1} + \cdots + c_n\frac{u_n'}{u_n}$$

with $c_1, \ldots, c_n \in \mathbb{C}$ and $v, u_1, \ldots, u_n \in F(t)$.

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with $c_1, \ldots, c_n \in \mathbb{C}$ and $v, u_1, \ldots, u_n \in F(t)$.
By logarithmic differentiation: the u_i's not in F are monic and irreducible in $F[t]$.

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By the lemma just one u_i is not in F and this u_i is t.

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with $c_1,\ldots,c_n \in \mathbb{C}$ and $v, u_1, \ldots, u_n \in F(t)$.
By the lemma just one u_i is not in F and this u_i is t.
So $c_1\frac{u_1'}{u_1} + \cdots + c_n\frac{u_n'}{u_n}$ is in F.

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$\int \frac{\sin x}{x}\,dz$, III

Finally, in

$$\frac{t^2-1}{tz} = v' + c_1\frac{u_1'}{u_1} + \cdots + c_n\frac{u_n'}{u_n}$$

we must have $v = \sum b_j t^j$ and from this: $\frac{1}{z} = b_1' + b_1$.