Elementarity and dimension

Non impeditus ab ulla scientia

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Milwaukee, 14 March, 2008: 10:15 – 10:35

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Outline

Covering dimension

Caveat: all spaces are (at least) normal

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Definition

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Definition

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

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Covering dimension

There is a convenient characterization.

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Theorem (Hemmingsen)

$\dim X \leqslant n$ *iff every* $n+2$*-element open cover has a shrinking with an empty intersection.*

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Large inductive dimension

Definition

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L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

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Dimensionsgrad

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The starting point: $\operatorname{Dg} X \leqslant -1$ iff $X = \emptyset$.

C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

(In)equalities

- For σ-compact metric X: $\dim X$

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(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X = \operatorname{Dg} X$
- The first equality is classical and holds for all metric X
- the second is fairly recent (1999).
- There is for each n a locally connected Polish X_n with $\operatorname{Dg} X = 1$ and $\dim X_n = n$ (Fedorchuk, van Mill)

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More inequalities

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- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)
- $\dim X \leqslant \operatorname{Dg} X$ (Fedorchuk)

We will reprove the last two inequalities.

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Definition

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These are (apparently) very rich substructures

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Examples

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How to make them

There are plenty of elementary substructures.

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How to make them

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Theorem (Löwenheim-Skolem)

Assume our language of discourse is countable. Let B be a structure suitable for that language and let $X \subseteq B$ then there is an elementary substructure A of B such that $X \subseteq A$ and $|A| \leqslant |X| + \aleph_0$.

Outline

1. Dimensions
2. Elementarity
3. Proofs using elementarity
   - Formulas
   - Bases
   - To work
4. Sources

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets

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$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2})\\
&\qquad\big[(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = 0) \rightarrow\\
&\qquad\big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \cdots \wedge (x_{n+2} \leqslant y_{n+2})\\
&\qquad\wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = 0)\\
&\qquad\wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1)\big)\big].
\end{aligned}
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$$(\forall x)(\forall y)(\exists u)$$
$$\Bigl[\Bigl(\bigl((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\bigr)\Bigr) \to \bigl(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\bigr)\Bigr]$$

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We start with $I_{-1}(a)$, which denotes $a = \mathsf{0}$

Dimensionsgrad

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$$(\forall x)(\forall y)(\exists u)$$
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Dimensionsgrad (auxiliary formulas)

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and $\operatorname{conn}(a)$ says that a is connected:

$$(\forall x)(\forall y)\bigl[\bigl((x \cap y = 0) \wedge (x \cup y = a)\bigr) \rightarrow \bigl((x = 0) \vee (x = a)\bigr)\bigr],$$

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Wherefore formulas?

Romeo and Juliet, Act 2, scene 2 (alternate)

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- $\operatorname{Ind} X \leqslant n$ iff 2^X satisfies $I_n(X)$
- $\operatorname{Dg} X \leqslant n$ iff 2^X satisfies $\Delta_n(X)$

Outline

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Covering dimension

Theorem

Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Covering dimension

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Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Proof.

Both directions use swelling and shrinking to replace the finite families by combinatorially equivalent subfamilies of the base. □

Large inductive dimension

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Proof.

Induction: given A and B expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between A' and B', such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As $\mathcal{B}_L$ is a base for the closed sets of L we know, by inductive assumption, that $\operatorname{Ind} L \leqslant n - 1$. □

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No equivalence, see later.

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Dimensionsgrad

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Proof.

Let $X = [0, 1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0, q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p, 1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).

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$\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously

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$\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously but $\operatorname{Dg}[0,1] = 1$. □

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Consider the Wallman space wL of L.

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Consider the Wallman space wL of L.

What can we say about $\dim wL$, $\operatorname{Ind} wL$ and $\operatorname{Dg} wL$?

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Proof.

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Notice that δ_n and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies δ_n iff L satisfies δ_n.

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Theorem

$\dim wL = \dim X$

Proof.

Notice that δ_n and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies δ_n iff L satisfies δ_n. Previous theorem: L satisfies δ_n iff 2^{wL} does.

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Covering dimension

Theorem

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Proof.

Notice that δ_n and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies δ_n iff L satisfies δ_n. Previous theorem: L satisfies δ_n iff 2^{wL} does. It follows that $\dim X \leqslant n$ iff $\dim wL \leqslant n$ for all n. □

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Notice that $I_n(a)$ and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies $I_n(X)$ iff L does.

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$\operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Proof.

Notice that $I_n(a)$ and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies $I_n(X)$ iff L does. By previous theorem we know $\operatorname{Ind} wL \leqslant n$, whenever L satisfies $I_n(wL)$.

Large inductive dimension

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$\operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Proof.

Notice that $I_n(a)$ and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies $I_n(X)$ iff L does. By previous theorem we know $\operatorname{Ind} wL \leqslant n$, whenever L satisfies $I_n(wL)$.
Thus: $\operatorname{Ind} X \leqslant n$ implies $\operatorname{Ind} wL \leqslant n$. $\square$

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Dimensionsgrad

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Notice that $\Delta_n(a)$ and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies $\Delta_n(X)$ iff L does.

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Nonroof

Notice that $\Delta_n(a)$ and its negation state that certain systems of equations have solutions. By elementarity we see that 2^X satisfies $\Delta_n(X)$ iff L does.

By previous theorem we know nothing yet about $\operatorname{Dg} wL$.

Dimensionsgrad

Theorem

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Dimensionsgrad

Theorem

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Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.

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Dimensionsgrad

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Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
There $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n-1$.

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Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
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Inductive assumption: $\operatorname{Dg} C \leqslant n-1$ in wL

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Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
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Inductive assumption: $\operatorname{Dg} C \leqslant n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.

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Proof.

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There $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n-1$.
Inductive assumption: $\operatorname{Dg} C \leqslant n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.
Still to show: C-in-wL is a cut between A and B in wL. □

Proof (continued)

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Back in X no component of H meets C, hence it does *not* meet both A and B.

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By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that

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Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

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Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.
Down in wL we have exactly the same relations, so H_A and H_B show F is not connected.

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Finishing up

Let X be compact Hausdorff and let L be a *countable* elementary sublattice of 2^X. Then

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There are X with $\dim X < \operatorname{Dg} X$, so $\operatorname{Dg} wL < \operatorname{Dg} X$ and $\operatorname{Ind} wL < \operatorname{Ind} X$ are possible.

Light reading

Website: `fa.its.tudelft.nl/~hart`

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On the Brouwer dimension of compact spaces, Mathematical Notes **73** (2003), 271–279,

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