Two nonimages of $\mathbb{H}^*$

Non impeditus ab ulla scientia

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Outline

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Čech-Stone compactification

Every completely regular space X has a compactification, βX, with the following extension property: if $f : X \to [0,1]$ is continuous then f admits an extension $\beta f : \beta X \to [0,1]$ (and hence also for arbitrary compact co-domains).

For normal X: whenever A and B are closed and disjoint *in* X then $\mathrm{cl}_{\beta X} A$ and $\mathrm{cl}_{\beta X} B$ are disjoint.

We write X^* to mean $\beta X \setminus X$.

$\beta\mathbb{N}$ and $\mathbb{N}^*$

$\mathbb{N}$ is the (discrete) space of natural numbers.

- $\beta\mathbb{N}$ is separable and extremally disconnected (disjoint open sets have disjoint closures)
- $\mathbb{N}^*$ is very not separable, very not extremally disconnected and also not self-Tietze

$\beta\mathbb{H}$ and $\mathbb{H}^*$

$\mathbb{H} = [0, \infty)$, the $\mathbb{H}$alf line.

- $\beta\mathbb{H}$ is separable and connected
- $\mathbb{H}^*$ is very not separable but it is connected
- it is also
  - one-dimensional
  - hereditarily unicoherent
    (if two continua meet their intersection is connected)
  - indecomposable (not the union of two proper subcontinua)

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What are the continuous images of $\mathbb{N}^*$ and $\mathbb{H}^*$?

Parovičenko: all compact spaces of weight $\aleph_1$ (or less) are continuous images of $\mathbb{N}^*$.

Dow and YT: all continua of weight $\aleph_1$ (or less) are continuous images of $\mathbb{H}^*$.

The Continuum Hypothesis (CH) implies:

- the continuous images of $\mathbb{N}^*$ are exactly the compact spaces of weight not more than $2^{\aleph_0}$
- the continuous images of $\mathbb{H}^*$ are exactly the continua of weight not more than $2^{\aleph_0}$

Isn't that a pretty parallel?

Separability and $\mathbb{N}^*$

Separable compact spaces have weight $2^{\aleph_0}$ (or less) but we can use 'β' to show

Theorem

Every separable compact space is an $\mathbb{N}^*$-image.

Proof.

Let D be countable and dense in the compact space X.
Let $f : \mathbb{N} \to D$ be onto with all fibers infinite.
Then $\beta f[\mathbb{N}^*] = X$ $\square$

So, no CH needed.

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Separability and $\mathbb{H}^*$

How about separable continua?
If K is one there is $f : \mathbb{N}^* \twoheadrightarrow K$.
Can we extend it to $F : \mathbb{H}^* \twoheadrightarrow K$?

One approach:

- embed K into $C = [0,1]^{2^{\aleph_0}}$
- extend f to $\bar{f} : \beta\mathbb{N} \to C$ (Tietze-Urysohn)
- connect the dots to get ...
- $F : \mathbb{H} \to C$ such that $\beta F[\mathbb{H}^*] = K$

Won't work, because ...

Separability and $\mathbb{H}^*$

... of the following

Example

Replicate the $\sin\frac{1}{x}$-curve along the positive real axis:

$$K_n = \{n\} \times [-1,1] \cup \Bigl\{\langle n+t, \sin\frac{\pi}{t}\rangle : 0 < t \leqslant 1\Bigr\}$$

Let $K = \bigcup_n K_n$.
The Open Colouring Axiom (OCA) implies that βK is not an $\mathbb{H}^*$-image.

Separability and $\mathbb{H}^*$

Very rough sketch of the argument

- Start with a putative $f : \mathbb{H}^* \twoheadrightarrow \beta K$
- Extract from it a continuous surjection $g : \mathbb{N}^* \twoheadrightarrow \beta\mathbb{N}$
- OCA implies g contains a similar map that is induced by a function $h : \mathbb{N} \to \mathbb{N}$
- Use h and the wiggly bits to create $s : \mathbb{N}^* \twoheadrightarrow \big(\omega \times (\omega + 1)\big)^*$
- OCA implies such surjections do not exist

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First-countability

Arkhangel′skiĭ's theorem(s)

A first-countable compact space has cardinality and hence(!) weight $2^{\aleph_0}$ or less.

Thus, CH implies first-countable compact spaces and continua are continuous images of $\mathbb{N}^*$ and $\mathbb{H}^*$, respectively.

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(A variation of) Bell's example

Example (Murray Bell)

There is a consistent example of a first-countable compact space that is not an $\mathbb{N}^*$-image.

Step 1

Add $\aleph_2$ many Cohen reals to your universe.
Use $\mathrm{Fn}(L, 2)$, where $L = \{\langle\alpha, \beta\rangle : \alpha < \beta < \omega_2\}$.
You get $E \subseteq L$ with the following property:
there is no family $\{A_\alpha : \alpha < \omega_2\}$ of subsets of $\mathbb{N}$ such that $\langle\alpha, \beta\rangle \in E$ if and only if $A_\alpha \cap A_\beta$ is infinite.

(A variation of) Bell's example

Step 2

Take the Alexandroff double, $\mathbb{I}$, of the unit interval:

- set: $[0,1] \times \{0,1\}$
- at $\langle x, 0\rangle$ basic neighbourhoods are
  $U(x,n) = \big((x - 2^{-n}, x + 2^{-n}) \times \{0,1\}\big) \setminus \{\langle x, 1\rangle\}$
- each $\langle x, 1\rangle$ isolated

(A variation of) Bell's example

Step 3

Work in $\mathbb{I}^2$.
For each $x \in [0,1]$ let C_x be the cross $\{\langle x, 1\rangle\} \times \mathbb{I} \cup \mathbb{I} \times \{\langle x, 1\rangle\}$

Step 4

Take an injection $\alpha \mapsto x_\alpha$ from ω_2 into $[0,1]$ and let
$E^s = \{\langle x_\alpha, x_\beta\rangle : \langle \alpha, \beta\rangle \in E \text{ or } \langle \alpha, \beta\rangle \in E\}$.
Delete from $\mathbb{I}^2$ all points $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle$ with $\langle x, y\rangle \notin E^s$.
What is left we call $\mathbb{I}_E$. Note: $\mathbb{I}_E$ is closed, hence compact.

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(A variation of) Bell's example

Step 5

Observe: $C_x \cap C_y \cap \mathbb{I}_E$ is nonempty if and only $\langle x, y\rangle \in E^s$.
Also: $C_x \cap \mathbb{I}_E$ is compact and open in $\mathbb{I}_E$.

Final step

Assume $f : \mathbb{N}^* \twoheadrightarrow \mathbb{I}_E$ is continuous and onto.
Choose, for each α, an infinite subset A_α of $\mathbb{N}$ such that
$f^{\leftarrow}[C_{x_\alpha}] = A_\alpha^*$
Now we have: $A_\alpha \cap A_\beta$ is infinite if and only if $\langle \alpha, \beta\rangle \in E$

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Our example

Example (Dow and YT)

There is a consistent example of a first-countable continuum that is not an $\mathbb{H}^*$-image.

Step 1

Add $\aleph_2$ many Cohen reals to your universe.
Use $\mathrm{Fn}(L, 2)$, where $L = \{\langle\alpha, \beta\rangle : \alpha < \beta < \omega_2\}$.
You get $E \subseteq L$ with the following property:
there is no family $\{U_\alpha : \alpha < \omega_2\}$ of open subset of $\mathbb{H}$ such that $\langle\alpha, \beta\rangle \in E$ if and only if $U_\alpha \cap U_\beta$ is unbounded

Our example

Step 2

Take the connected comb $\mathbb{C}$, Saalfrank's connected version of the Alexandroff double of the unit interval:

- set: $[0,1] \times [0,1]$
- at $\langle x, 0\rangle$ basic neighbourhoods are $U(x,n) = \bigl((x-2^{-n}, x+2^{-n}) \times [0,1]\bigr) \setminus \bigl(\{x\} \times [2^{-n}, 1]\bigr)$
- at $\langle x, y\rangle$ $(y > 0)$ basic neighbourhoods are $U(x,y,n) = \{x\} \times (y-2^{-n}, y+2^{-n})$

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Our example

Step 3

Work in $\mathbb{C}^2$.
For each $x \in [0,1]$ let C_x be the (two-dimensional) cross

$$\bigl(\{x\} \times (0,1] \times \mathbb{C}\bigr) \cup \bigl(\mathbb{C} \times \{x\} \times (0,1]\bigr)$$

Step 4

Take an injection $\alpha \mapsto x_\alpha$ from ω_2 into $[0,1]$ and let
$E^s = \{\langle x_\alpha, x_\beta\rangle : \langle \alpha, \beta\rangle \in E \text{ or } \langle \alpha, \beta\rangle \in E\}$.

Our example

Step 5

Delete from $\mathbb{C}^2$ all open sets

$$\{x\} \times (0,1] \times \{y\} \times (0,1]$$

for which $\langle x, y\rangle \notin E^s$.
What is left we call $\mathbb{C}_E$. Note: $\mathbb{C}_E$ is closed, hence compact.
The space $\mathbb{C}_E$ is also (arcwise) connected.

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Our example

Step 6

Observe: $C_x \cap C_y \cap \mathbb{C}_E$ is nonempty if and only $\langle x, y\rangle \in E^s$.
Also: $C_x \cap \mathbb{C}_E$ is open in $\mathbb{C}_E$ (but not clopen).

Step 7

We need substitutes for the clopen sets.
For $x \in [0,1]$ and $a > 0$ let
$D_{x,a} = \bigl(\{x\} \times [a,1] \times \mathbb{C}\bigr) \cup \bigl(\mathbb{C} \times \{x\} \times [a,1]\bigr)$.
Note: if $a < b$ then $D_{x,b}$ is contained in the interior of $D_{x,a}$.

Our example

Final step

Assume $f : \mathbb{H}^* \twoheadrightarrow \mathbb{C}_E$ is continuous and onto.
Choose, for each α, an open subset U_α of $\mathbb{H}$ such that

$$f^{\leftarrow}[D_{x_\alpha,1}] \subseteq \operatorname{Ex} U_\alpha \subseteq f^{\leftarrow}[D_{x_\alpha,\frac{1}{2}}]$$

Now we have: $U_\alpha \cap U_\beta$ is unbounded if and only if $\langle \alpha, \beta \rangle \in E$ (because the same holds for the families $\{D_{x,a}\}_x$).

Light reading

Website: `http://fa.its.tudelft.nl/~hart`

Alan Dow and Klaas Pieter Hart,
A separable non-remainder of $\mathbb{H}$,
`http://arxiv.org/abs/0805.2265`

Alan Dow and Klaas Pieter Hart,
A first-countable non-remainder of $\mathbb{H}$,
`http://arxiv.org/abs/0708.4739`