The Löwenheim-Skolem theorem has been very good to me Non impeditus ab ulla scientia

K. P. Hart

Faculty EEMCS TU Delft

Paris, 10 November, 2008: 14:00-15:15







2 Dimensions









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2 Dimensions

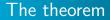
3 Categoricity

A problem of Lelek

5 Sources

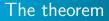


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Just so we know what we are talking about.



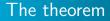


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Theorem

Let A be a structure for some language \mathcal{L} and let X be a subset of A. There is an elementary substructure B of A of cardinality at most $\aleph_0 \cdot |X| \cdot |\mathcal{L}|$.





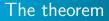
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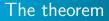
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We consider two languages:

- lattice theory
- set theory





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- We take the Wallman space, wL, of L.
- wL is compact metrizable and looks a lot like X.
- We can use wL to get information about X.
- Or reduce general problems to the metrizable case.







Wallman space

It's like the Stone space of a Boolean algebra.

• Underlying set of *wL*: the ultrafilters on *L*.



Wallman space

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- For $a \in L$ put $\bar{a} = \{u \in wL : a \in u\}$.
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- The map $q_L : x \mapsto \{a \in L : x \in a\}$ is a continuous surjection.



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- $\{\bar{a} : a \in L\}$ serves as a base for the closed sets of wL.
- The map $q_L : x \mapsto \{a \in L : x \in a\}$ is a continuous surjection.
- Because *L* is countable, the space *wL* is metrizable









- 3 Categoricity
- A problem of Lelek





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Covering dimension

Definition

 $\dim X \le n$ if every finite open cover has a (finite) open refinement of order at most n+1



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There is a convenient first-order characterization.

Theorem (Hemmingsen)

dim $X \le n$ iff every n + 2-element open cover has a shrinking with an empty intersection.



Large inductive dimension

Definition

Ind $X \le n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\text{Ind } L \le n - 1$.



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L is a partition between *A* and *B* means: there are closed sets *F* and *G* that cover *X* and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.



Dimensionsgrad

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 $\operatorname{Dg} X \leq n$ between every two disjoint closed sets A and B there is a cut C that satisfies $\operatorname{Dg} C \leq n-1$.



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C is a cut between *A* and *B* means: $C \cap K \neq \emptyset$ whenever *K* is a subcontinuum of *X* that meets both *A* and *B*.



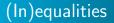


• For σ -compact metric X: dim X



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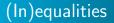
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• For σ -compact metric X: dim $X = \operatorname{Ind} X$



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- For σ-compact metric X: dim X = Ind X = Dg X
- The first equality is classical and holds for all metric X
- the second is fairly recent (1999).
- There is for each *n* a locally connected Polish X_n with $\operatorname{Dg} X_n = 1$ and dim $X_n = n$ (Fedorchuk, van Mill)



More inequalities

For arbitrary compact Hausdorff spaces



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For arbitrary compact Hausdorff spaces:

• $Dg X \leq Ind X$ (each partition is a cut)



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- dim $X \leq \operatorname{Ind} X$ (Vedenissof)
- dim X ≤ Dg X (Fedorchuk)



More inequalities

For arbitrary compact Hausdorff spaces:

- $Dg X \leq Ind X$ (each partition is a cut)
- dim $X \leq \operatorname{Ind} X$ (Vedenissof)
- dim $X \leq Dg X$ (Fedorchuk)

We will reprove the last two inequalities.



Covering dimension

Here is Hemmingsen's characterization of dim $X \leq n$



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Covering dimension

Here is Hemmingsen's characterization of dim $X \leq n$ reformulated in terms of closed sets



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Covering dimension

Here is Hemmingsen's characterization of dim $X \le n$ reformulated in terms of closed sets and cast as a formula, δ_n , in the language of lattices



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Covering dimension

Here is Hemmingsen's characterization of dim $X \le n$ reformulated in terms of closed sets and cast as a formula, δ_n , in the language of lattices

$$(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2}) \\ [(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = \mathbf{0}) \implies \\ ((x_1 \leq y_1) \wedge (x_2 \leq y_2) \wedge \cdots \wedge (x_{n+2} \leq y_{n+2}) \\ \wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = \mathbf{0}) \\ \wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = \mathbf{1}))].$$

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We can express $\operatorname{Ind} X \leq n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)



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 $(\forall x)(\forall y)(\exists u)$ $[(((x \le a) \land (y \le a) \land (x \cap y = o)) \implies (partn(u, x, y, a) \land I_{n-1}(u))]$



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We start with $I_{-1}(a)$, which denotes a = o

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Here we have the recursive definition of a formula $\Delta_n(a)$:



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Here we have the recursive definition of a formula $\Delta_n(a)$:

$$\begin{aligned} &(\forall x)(\forall y)(\exists u)\\ &\left[\left((x \leq a) \land (y \leq a) \land (x \cap y = o)\right) \implies (\operatorname{cut}(u, x, y, a) \land \Delta_{n-1}(u))\right],\\ &\text{and } \Delta_{-1}(a) \text{ denotes } a = o. \end{aligned}$$



Dimensionsgrad (auxiliary formulas)

The formula cut(u, x, y, a) expresses that u is a cut between x and y in a:



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The formula cut(u, x, y, a) expresses that u is a cut between x and y in a:

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and conn(a) says that a is connected:

$$(\forall x)(\forall y)[((x \cap y = 0) \land (x \cup y = a)) \implies ((x = 0) \lor (x = a))],$$



Why formulas?

• dim
$$X \leq n$$
 iff $2^X \models \delta_n$



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Covering dimension

Theorem

Let X be compact. Then dim $X \le n$ iff some (every) lattice-base for its closed sets satisfies δ_n .



Covering dimension

Theorem

Let X be compact. Then dim $X \le n$ iff some (every) lattice-base for its closed sets satisfies δ_n .

Proof.

Both directions use swelling and shrinking to replace the finite families by combinatorially equivalent subfamilies of the base.



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Let X be compact. If some lattice lattice-base, \mathcal{B} , for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leq n$.



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Proof.

Induction: given A and B expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between A' and B', such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As \mathcal{B}_L is a base for the closed sets of L we know, by inductive assumption, that $\operatorname{Ind} L \leq n-1$.



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No equivalence, see later.



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Proof.

Let X = [0, 1] and let \mathcal{B} be the lattice-base generated by the family of sets of the form $[0, q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p, 1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).

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By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$.



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By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does.



Covering dimension

Theorem

 $\dim wL = \dim X$

Proof.

By elementarity we see that $2^X \models \delta_n$ iff $L \models \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does. It follows that dim $X \le n$ iff dim $wL \le n$ for all n.



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Proof.

By elementarity we see that $2^X \vDash I_n(X)$ iff $L \vDash I_n(X)$.



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Ind $wL \leq \operatorname{Ind} X$

Proof.

By elementarity we see that $2^X \models I_n(X)$ iff $L \models I_n(X)$. By previous theorem we know $\text{Ind } wL \le n$, whenever L satisfies $I_n(wL)$.



Large inductive dimension

Theorem

Ind $wL \leq \operatorname{Ind} X$

Proof.

By elementarity we see that $2^X \models I_n(X)$ iff $L \models I_n(X)$. By previous theorem we know $\text{Ind } wL \le n$, whenever L satisfies $I_n(wL)$. Thus: $\text{Ind } X \le n$ implies $\text{Ind } wL \le n$.



Dimensionsgrad

Theorem

 $\mathsf{Dg} \, wL \leq \mathsf{Dg} \, X$



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Theorem

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Theorem

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Nonproof

By elementarity we see that $2^X \models \Delta_n(X)$ iff $L \models \Delta_n(X)$. By previous theorem we know nothing yet about Dg *wL*.



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 $\operatorname{Dg} wL \leq \operatorname{Dg} X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.



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Dimensionsgrad

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Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$. Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leq n-1$.

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Theorem

 $\mathsf{Dg} \, \mathsf{wL} \leq \mathsf{Dg} \, \mathsf{X}$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$. Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leq n-1$. Inductive assumption: Dg $C \leq n-1$ in wL

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 $\mathsf{Dg} \, \mathsf{wL} \leq \mathsf{Dg} \, X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$. Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leq n-1$. Inductive assumption: $Dg C \leq n-1$ in wL, because $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM. Still to show: C-in-wL is a cut between A and B in wL.

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Proof (continued)

Let F be a closed set in wL that meets A and B but not C.



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Proof (continued)

Let F be a closed set in wL that meets A and B but not C. We show F is not connected.



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Proof (continued)

Let F be a closed set in wL that meets A and B but not C. We show F is not connected. Find H in L around F, disjoint from C.



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Proof (continued)

Let F be a closed set in wL that meets A and B but not C. We show F is not connected. Find H in L around F, disjoint from C. Back in X no component of H meets C, hence it does *not* meet both A and B.



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By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

Down in wL we have exactly the same relations, so H_A and H_B show F is not connected.





Let X be compact Hausdorff and let L be a *countable* elementary sublattice of 2^X . Then





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Finishing up

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There are X with dim X < Dg X, so Dg wL < Dg X and Ind wL < Ind X are possible.







2 Dimensions









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Given a metric continuum \boldsymbol{X} there is another metric continuum \boldsymbol{Y} such that





Given a metric continuum X there is another metric continuum Y such that

• X and Y look the same (they have elementarily equivalent countable bases)





Given a metric continuum X there is another metric continuum Y such that

- X and Y look the same (they have elementarily equivalent countable bases)
- X and Y are not homeomorphic



Example: zero-dimensionality

Here is a first-order sentence, call it $\boldsymbol{\zeta}$

$$(\forall x)(\forall y)(\exists u)(\exists v) ((x \sqcap y = o) \to ((x \le u) \land (y \le v) \land (u \sqcap v = o) \land (u \sqcup v = 1)))$$



Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v) ((x \sqcap y = 0) \to ((x \le u) \land (y \le v) \land (u \sqcap v = 0) \land (u \sqcup v = 1)))$$

In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).



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By *compactness*, if some base satisfies this sentence then the space is zero-dimensional.



Example: no isolated points

Here is a another first-order sentence, call it $\boldsymbol{\pi}$

$$(\forall x)(\exists y)((x < 1) \rightarrow ((x < y) \land (y < 1)))$$



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If some base satisfies this sentence then the space has no isolated points.



Example: the Cantor set is categorical

Let X be compact metric with a countable base \mathcal{B} for the closed sets that satisfies ζ and π .



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Thus: if X looks like C then X is homeomorphic to C.

The Cantor set is categorical among compact metric spaces.



What the main result says

Among metric continua there is no categorical space.



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What the main result says

Among metric continua there is no categorical space. No (in)finite list of first-order properties will characterize a single metric continuum.



A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is



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The pseudoarc is the only metric continuum that is

• hereditarily indecomposable and



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The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and
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A two-item list but Chainability is *not* first-order.



A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

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A two-item list but ... Chainability is *not* first-order. (Hereditary indecomposability is.)



An embedding lemma

Lemma

Let X and Z be metric continua, with countable lattice bases, \mathcal{B} and \mathcal{C} , for their respective families of closed sets. Let u be a free ultrafilter on ω . There is an embedding of \mathcal{C} into the ultrapower of \mathcal{B} by u.



How to make Y

Let X and Z be metric continua, with countable lattice bases, \mathcal{B} and \mathcal{C} , for their respective families of closed sets.



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Apply the Löwenheim-Skolem theorem:



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Apply the Löwenheim-Skolem theorem: Find a countable elementary sublattice \mathcal{D} of \mathcal{B}_u that contains $\varphi[\mathcal{C}]$.



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Apply the Löwenheim-Skolem theorem: Find a countable elementary sublattice \mathcal{D} of \mathcal{B}_u that contains $\varphi[\mathcal{C}]$. Let Y be the Wallman space of \mathcal{D} .





• Y is compact metric (\mathcal{D} is countable).



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- \mathcal{D} is a base for the closed sets of Y (by Wallman's theorem).
- \mathcal{D} is elementarily equivalent to \mathcal{B}_u and hence to \mathcal{B} .
- Y maps onto Z (because $\varphi[\mathcal{C}]$ is embedded into \mathcal{D}).



Getting a good Y

Let X be given, with a countable base \mathcal{B} for its closed sets.



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Let X be given, with a countable base \mathcal{B} for its closed sets. There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).



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Getting a good Y

Let X be given, with a countable base \mathcal{B} for its closed sets. There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).

Find Y with a base that is elementarily equivalent to \mathcal{B} and such that Y maps onto Z.

So: Y is not homeomorphic to X.







2 Dimensions

3 Categoricity







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Chainability

Definition

A continuum, X, is chainable if every (finite) open cover \mathcal{U} has an open chain-refinement \mathcal{V} .



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A continuum, X, is chainable if every (finite) open cover \mathcal{U} has an open chain-refinement \mathcal{V} . \mathcal{V} can be written as $\{V_i : i < n\}$ such that $V_i \cap V_j \neq \emptyset$ iff $|i-j| \leq 1$.



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[0,1] is chainable; the circle S^1 is not.



Span zero

Definition

A continuum, X, has xxx span zero if every subcontinuum Z of $X \times X$ that satisfies yyy intersects the diagonal $\{\langle x, x \rangle : x \in X\}$.



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 $\begin{array}{ccc} \mathsf{x}\mathsf{x}\mathsf{x} & \mathsf{y}\mathsf{y}\mathsf{y}\\ \ldots & \pi_1[Z] = \pi_2[Z] \end{array}$



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$$\begin{array}{ll} \mathsf{xxx} & \mathsf{yyy} \\ \dots & \pi_1[Z] = \pi_2[Z] \\ \mathsf{semi} & \pi_1[Z] \subseteq \pi_2[Z] \end{array}$$



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xxx semi surjective

yyy

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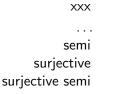


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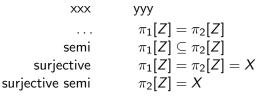
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[0,1] has all spans zero, S^1 has all spans non-zero





Theorem

In a chainable continuum all spans are zero.



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Question (Lelek)

What about the converse?





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This is an important problem in metric continuum theory.





Theorem

In a chainable continuum all spans are zero.

Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory. We free it from the metric constraints.



Reflection

Theorem

Any counterexample to Lelek's problem can be converted into a metrizable counterexample.



Reflection

Theorem

Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

Proof.

Let X be a counterexample, let $L \prec 2^X$. Then wL is a metrizable counterexample.



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Not quite ...



Complications

(Non-)chainability is not a firt-order property of the lattice 2^{X} .



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Their natural formulations are $L_{\omega_1,\omega}$ -formulas.



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Chainability:

$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)((u_1\cup u_2\cup u_3\cup u_4=X)\rightarrow \bigvee_{n\in\omega}\Phi_n(u_1,u_2,u_3,u_4))$$



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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an *n*-element chain refinement.



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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an *n*-element chain refinement.

It suffices to consider four-element open covers only.

Solution: Use Set Theory

Let θ be 'suitably large' and let $M \prec H(\theta)$ be a countable elementary substructure



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- wL has span zero iff X has span zero (any kind)



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Theorem

In this situation:

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- wL has span zero iff X has span zero (any kind)

Proof for Chainability.

Chainability is now first-order and, like covering dimension, one needs only consider covers and refinements that belong to a certain base. $\hfill \Box$



Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$.



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This gives the easy part: if there is a 'bad' continuum in $X \times X$ then there is one in M and it is equally bad in $wL \times wL$.



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For the converse ...



Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.



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Span zero, continued

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Easier said than constructed



Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK} .



Span zero, the real argument

Apply Shelah's Ultrapower theorem



Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal κ , an ultrafilter u on κ and an isomorphism $h: \prod_{u} (2^{X \times X}) \to \prod_{u} wK$ (which can be taken to be the identity on K).



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How does that help?



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How does that help?

For that we need some topology.



Dualizing ultrapowers

Take a compact Hausdorff space Y with a lattice base B. Also take a cardinal κ and an ultrafilter u on κ .



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The Wallman space of the ultrapower $\prod_{u} B$ is the fiber $p_{\kappa}^{\leftarrow}(u)$. Bankston calls this the ultracopower of Y; we write Y_{u} .



Span zero, the real argument



Span zero, the real argument

Back to
$$Z \subseteq wK$$
.
• Let $Z_u = cl(\kappa \times Z) \cap p_{\kappa}^{\leftarrow}(u)$.



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Span zero, the real argument

- Let $Z_u = \operatorname{cl}(\kappa \times Z) \cap p_{\kappa}^{\leftarrow}(u)$.
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Span zero, the real argument

- Let $Z_u = \operatorname{cl}(\kappa \times Z) \cap p_{\kappa}^{\leftarrow}(u)$.
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- $wh[Z_u]$ is a continuum in $(X \times X)_u$ (wh is dual to h).



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And

$$q_{\mathcal{K}}[Z_X] = q_{\mathcal{K}}\Big[p_{X \times X}\big[wh[Z_u]\big]\Big] = p_{w\mathcal{K}}\Big[(wh)^{-1}\big[wh[Z_u]\big]\Big] = Z$$



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So, that's it!?

Span zero, the real argument

Back to $Z \subseteq wK$.

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- $Z_X = p_{X \times X} [wh[Z_u]]$ is a continuum in $X \times X$.

And

$$q_{\mathcal{K}}[Z_X] = q_{\mathcal{K}}\Big[p_{X \times X}\big[wh[Z_u]\big]\Big] = p_{w\mathcal{K}}\Big[(wh)^{-1}\big[wh[Z_u]\big]\Big] = Z$$

So, that's it!? Almost.

Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2 .



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Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.



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Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.







2 Dimensions

3 Categoricity

A problem of Lelek





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Light reading

Website: fa.its.tudelft.nl/~hart

🔋 K. P. Hart.

Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.

📕 K. P. Hart.

There is no categorical metric continuum, to appear.

D. Bartošová, K. P. Hart, B. van der Steeg, Lelek's problem is not a metric problem, to appear.

