The Löwenheim-Skolem theorem has been very good to me

Non impeditus ab ulla scientia

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Outline

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The theorem

Just so we know what we are talking about.

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Let A be a structure for some language $\mathcal{L}$ and let X be a subset of A. There is an elementary substructure B of A of cardinality at most $\aleph_0 \cdot |X| \cdot |\mathcal{L}|$.

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Why lattices?

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- wL is compact metrizable and looks a lot like X.
- We can use wL to get information about X.
- Or reduce general problems to the metrizable case.

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- The map $q_L : x \mapsto \{a \in L : x \in a\}$ is a continuous surjection.
- Because L is countable, the space wL is metrizable

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Theorem (Hemmingsen)

$\dim X \leq n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

Large inductive dimension

Definition

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L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

Dimensionsgrad

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C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

(In)equalities

- For σ-compact metric X: $\dim X$

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(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X = \operatorname{Dg} X$
- The first equality is classical and holds for all metric X
- the second is fairly recent (1999).
- There is for each n a locally connected Polish X_n with $\operatorname{Dg} X_n = 1$ and $\dim X_n = n$ (Fedorchuk, van Mill)

More inequalities

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For arbitrary compact Hausdorff spaces:

- $\operatorname{Dg} X \leq \operatorname{Ind} X$ (each partition is a cut)
- $\dim X \leq \operatorname{Ind} X$ (Vedenissof)
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We will reprove the last two inequalities.

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$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2})\\
&\qquad\Big[(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = 0) \implies\\
&\qquad\big((x_1 \le y_1) \wedge (x_2 \le y_2) \wedge \cdots \wedge (x_{n+2} \le y_{n+2})\\
&\qquad\wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = 0)\\
&\qquad\wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1)\big)\Big].
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$$\Big[\Big(\big((x \le a) \wedge (y \le a) \wedge (x \cap y = \mathsf{0})\big) \implies \big(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\big)\Big)\Big]$$

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We start with $I_{-1}(a)$, which denotes $a = 0$

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Covering dimension

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Let X be compact. Then $\dim X \leq n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Proof.

Both directions use swelling and shrinking to replace the finite families by combinatorially equivalent subfamilies of the base. □

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No equivalence, see later.

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Proof.

Let $X = [0,1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0,q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p,1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).

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$\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously but $\operatorname{Dg}[0,1] = 1$. $\square$

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Proof.

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By elementarity we see that $2^X \models \delta_n$ iff $L \models \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does.

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By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does.

It follows that $\dim X \leq n$ iff $\dim wL \leq n$ for all n. $\square$

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Proof.

By elementarity we see that $2^X \vDash I_n(X)$ iff $L \vDash I_n(X)$. By previous theorem we know $\operatorname{Ind} wL \leq n$, whenever L satisfies $I_n(wL)$.

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Proof.

By elementarity we see that $2^X \vDash I_n(X)$ iff $L \vDash I_n(X)$. By previous theorem we know $\operatorname{Ind} wL \leq n$, whenever L satisfies $I_n(wL)$. Thus: $\operatorname{Ind} X \leq n$ implies $\operatorname{Ind} wL \leq n$. $\square$

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By elementarity we see that $2^X \vDash \Delta_n(X)$ iff $L \vDash \Delta_n(X)$.
By previous theorem we know nothing yet about $\operatorname{Dg} wL$.

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Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leq n-1$.

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Inductive assumption: $\operatorname{Dg} C \leq n-1$ in wL

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Inductive assumption: $\operatorname{Dg} C \le n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.

Dimensionsgrad

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Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \le n-1$.
Inductive assumption: $\mathrm{Dg}\, C \le n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.
Still to show: C-in-wL is a cut between A and B in wL. □

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Proof (continued)

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By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

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Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.
Down in wL we have exactly the same relations, so H_A and H_B show F is not connected.

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Finishing up

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There are X with $\dim X < \operatorname{Dg} X$, so $\operatorname{Dg} wL < \operatorname{Dg} X$ and $\operatorname{Ind} wL < \operatorname{Ind} X$ are possible.

Outline

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The result

Given a metric continuum X there is another metric continuum Y such that

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  (they have elementarily equivalent countable bases)

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Given a metric continuum X there is another metric continuum Y such that

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  (they have elementarily equivalent countable bases)
- X and Y are not homeomorphic

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\bigl((x \sqcap y = 0) \to ((x \leq u) \land (y \leq v) \land (u \sqcap v = 0) \land (u \sqcup v = 1))\bigr)$$

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In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).
By *compactness*, if some base satisfies this sentence then the space is zero-dimensional.

Example: no isolated points

Here is a another first-order sentence, call it π

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In words: every closed proper subset (x) is properly contained in a closed proper subset (y);
in fewer words: there are no isolated points.
If some base satisfies this sentence then the space has no isolated points.

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Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.

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Thus: if X looks like C then X is homeomorphic to C.

The Cantor set is categorical among compact metric spaces.

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What the main result says

Among metric continua there is no categorical space.

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Among metric continua there is no categorical space.
No (in)finite list of first-order properties will characterize a single metric continuum.

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

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Chainability is *not* first-order.

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A two-item list but . . .
Chainability is *not* first-order.
(Hereditary indecomposability is.)

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An embedding lemma

Lemma

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
There is an embedding of $\mathcal{C}$ into the ultrapower of $\mathcal{B}$ by u.

How to make Y

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Let $\varphi : \mathcal{C} \to \mathcal{B}_u$ be an embedding.

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Find a countable elementary sublattice $\mathcal{D}$ of $\mathcal{B}_u$ that contains $\varphi[\mathcal{C}]$.

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Let Y be the Wallman space of $\mathcal{D}$.

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Properties of Y

- Y is compact metric ($\mathcal{D}$ is countable).

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- $\mathcal{D}$ is a base for the closed sets of Y (by Wallman's theorem).
- $\mathcal{D}$ is elementarily equivalent to $\mathcal{B}_u$ and hence to $\mathcal{B}$.
- Y maps onto Z (because $\varphi[\mathcal{C}]$ is embedded into $\mathcal{D}$).

Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets.

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There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).
Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and such that Y maps onto Z.
So: Y is not homeomorphic to X.

Outline

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Definition

A continuum, X, is chainable if every (finite) open cover $\mathcal{U}$ has an open chain-refinement $\mathcal{V}$.

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$[0, 1]$ is chainable; the circle S^1 is not.

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Span zero

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A continuum, X, has xxx span zero if every subcontinuum Z of $X \times X$ that satisfies yyy intersects the diagonal $\{\langle x, x\rangle : x \in X\}$.

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$[0, 1]$ has all spans zero, S^1 has all spans non-zero

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The problem

Theorem

In a chainable continuum all spans are zero.

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Question (Lelek)

What about the converse?

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This is an important problem in metric continuum theory.

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Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory.
We free it from the metric constraints.

Reflection

Theorem

Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

Reflection

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Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

Proof.

Let X be a counterexample, let $L \prec 2^X$. Then wL is a metrizable counterexample. □

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Not quite ...

Complications

(Non-)chainability is not a firt-order property of the lattice 2^X.

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Chainability:

$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)\Big((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \to \bigvee_{n\in\omega} \Phi_n(u_1, u_2, u_3, u_4)\Big)$$

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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an n-element chain refinement.

It suffices to consider four-element open covers only.

Solution: Use Set Theory

Let θ be 'suitably large' and let $M \prec H(\theta)$ be a countable elementary substructure

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- wL *has span zero iff* X *has span zero (any kind)*

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Theorem

In this situation:

- *wL is chainable iff X is chainable*
- *wL has span zero iff X has span zero (any kind)*

Proof for Chainability.

Chainability is now first-order and, like covering dimension, one needs only consider covers and refinements that belong to a certain base. □

Span zero

Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$.

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For the converse ...

Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

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Easier said than constructed

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Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK}.

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Span zero, the real argument

Apply Shelah's Ultrapower theorem

Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal κ, an ultrafilter u on κ and an isomorphism $h : \prod_u(2^{X\times X}) \to \prod_u wK$ (which can be taken to be the identity on K).

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How does that help?

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How does that help?

For that we need some topology.

Dualizing ultrapowers

Take a compact Hausdorff space Y with a lattice base B. Also take a cardinal κ and an ultrafilter u on κ.

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The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_\kappa^{\leftarrow}(u)$. Bankston calls this the ultracopower of Y; we write Y_u.

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Back to $Z \subseteq wK$.

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- Let $Z_u = \mathrm{cl}(\kappa \times Z) \cap p_\kappa^\leftarrow(u)$.

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- Let $Z_u = \mathrm{cl}(\kappa \times Z) \cap p_\kappa^{\leftarrow}(u)$.
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- $wh[Z_u]$ is a continuum in $(X \times X)_u$ (wh is dual to h).

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- $Z_X = p_{X \times X}\left[wh[Z_u]\right]$ is a continuum in $X \times X$.

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- And

$$q_K[Z_X] = q_K\Bigl[p_{X \times X}\bigl[wh[Z_u]\bigr]\Bigr] = p_{wK}\Bigl[(wh)^{-1}\bigl[wh[Z_u]\bigr]\Bigr] = Z$$

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So, that's it!?

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So, that's it!? Almost.

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Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2.

Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2.

Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.

Outline

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Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart.
Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.

K. P. Hart.
There is no categorical metric continuum, to appear.

D. Bartošová, K. P. Hart, B. van der Steeg,
Lelek's problem is not a metric problem, to appear.

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