A tale of three Boolean algebras

Non impeditus ab ulla scientia

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Outline

Borel sets and ideals

Our starting point is $\mathcal{B}$, the σ-algebra of Borel sets of $[0,1]$.

It has two natural ideals:

- $\mathcal{M}$, the sets of first category
- $\mathcal{N}$, the sets of measure zero

Ideal: closed under finite unions and taking subsets.
$\mathcal{M}$ and $\mathcal{N}$ are even σ-ideals: closed under countable unions.

Category algebra

The Category algebra, $\mathbb{C}$, is the quotient algebra $\mathcal{B}/\mathcal{M}$.
That is, $A \sim B$ iff $A \triangle B \in \mathcal{M}$

It is isomorphic to the regular open algebra of $[0,1]$.
This follows from

- $\{A : A \text{ is equivalent to an open set}\}$ is a σ-algebra, hence
- every Borel set is equivalent to an open set
- if O is open then it is equivalent to $\operatorname{int}\operatorname{cl} O$
- a regular open set — $O = \operatorname{int}\operatorname{cl} O$ — is only equivalent to itself

Operations: $U \vee V = \operatorname{int}\operatorname{cl}(U \cup V)$, $U \wedge V = U \cap V$ and $U' = [0,1] \setminus \operatorname{cl} U$.

Measure algebra

The Measure algebra, $\mathbb{M}$, is the quotient algebra $\mathcal{B}/\mathcal{N}$.
That is, $A \sim B$ iff $A \triangle B \in \mathcal{N}$

It is not isomorphic to any easliy recognizable algebra.

Unless you know $\mathbb{M}$ very well, in which case ...

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$\mathcal{P}(\mathbb{N})/\mathit{fin}$

This is clearly a quotient: the power set of $\mathbb{N}$ modulo the ideal, *fin*, of finite sets.

Some common notations in this case

- $A =^* B$ for $A \triangle B \in \mathit{fin}$
- $A \subseteq^* B$ for $A \setminus B \in \mathit{fin}$
- etcetera

Also $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is not isomorphic to an easily recognizable algebra, unless of course ...

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They are familiar, though

The algebras correspond to three ‘familiar’ objects from Fucntional Analysis.

- $\mathbb{C}$ goes with the Dedekind completion of $C\big([0,1]\big)$
- $\mathbb{M}$ goes with $L_\infty[0,1]$
- $\mathcal{P}(\mathbb{N})/\mathit{fin}$ goes with ℓ_∞/c_0

We’ll see how that works later.

Tarski’s theorem

All three algebras are atomless.

This means that they are elementarily equivalent.
That is, they satisfy the exact same Boolean algebraic sentences.

This was proved by Tarski in the 1930’s.

Separability, etc

Theorem

The algebra $\mathbb{C}$ is the union of countably many filters:

For each rational q the family

$$\mathcal{F}_q = \{U : q \in U\}$$

is a filter and

$$\mathbb{C} = \bigcup \{\mathcal{F}_q : q \in \mathbb{Q} \cap [0,1]\}$$

We say $\mathbb{C}$ is σ-centered

Separability, etc

Theorem

The algebra $\mathbb{M}$ is not σ-centered.

Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a family of filters in $\mathbb{M}$; we assume they are ultrafilters.

For each n pick B_n in $\mathcal{F}_n$ of measure less that 3^{-n}.
(Think of the intervals $[i3^{-n}, (i+1)3^{-n})$ for $i < 3^n$.)

The union $\bigcup_n B_n$ has measure at most $\frac{1}{2}$; its complement belongs to no $\mathcal{F}_n$.

Separability, etc

Theorem

The algebra $\mathbb{M}$ is σ-linked.

For each rational interval $[p, q)$ put

$$\mathcal{L}_{p,q} = \{B : \mu(B \cap [p, q)) > \frac{1}{2}(q - p)\}$$

if $B_1, B_2 \in \mathcal{L}_{p,q}$ then $\mu(B_1 \cap B_2) > 0$, so $\mathcal{L}_{p.q}$ is linked.

$\mathbb{M}$ is the union of the countably many $\mathcal{L}_{p,q}$.

The three algebras
No difference
Large differences
Topology
Other relations
Embedding into $\mathcal{P}(\mathbb{N})/fin$

Separability, etc

Theorem

The algebra $\mathcal{P}(\mathbb{N})/fin$ has a pairwise disjont subset of size $|\mathbb{R}|$.

Sierpiński: if x is irrational and larger than 1 put

$$S_x = \{\frac{1}{n}\lfloor nx \rfloor : n \in \mathbb{N}\}$$

- S_x is (the range of) a sequence of rational numbers that converges to x.
- if $x \neq y$ then $S_x \cap S_y =^* \emptyset$.
- so $\{S_x : x > 0 \text{ and } x \text{ irrational}\}$ gives the set we seek.

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Stone space

If B is a Boolean algebra then we let $\mathrm{St}(B)$ denote the set of ultrafilters on B.

For $a \in B$ put $\bar{a} = \{u \in \mathrm{St}(B) : a \in u\}$

$\{\bar{a} : a \in B\}$ is a base for a topology on $\mathrm{St}(B)$ that is zero-dimensional compact Hausdorff and has $\{\bar{a} : a \in B\}$ as its family of clopen sets.

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Three spaces

- $\mathrm{St}(\mathbb{C})$ is separable
- $\mathrm{St}(\mathbb{M})$ is not separable but it satisfies the countable chain condition
- $\mathrm{St}(\mathcal{P}(\mathbb{N})/\mathit{fin})$ does not satisfy the countable chain condition

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Three Banach algebras

As promised:

- $C\big(\mathrm{St}(\mathbb{C})\big)$ is the Dedekind completion of $C\big([0,1]\big)$
- $C\big(\mathrm{St}(\mathbb{M})\big)$ is $L_\infty[0,1]$
- $C\big(\mathrm{St}(\mathcal{P}(\mathbb{N})/\mathit{fin})\big)$ is ℓ_∞/c_0

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Embeddability

We investigate mutual embeddability of the algebras.
Useful observation:

Theorem

B is isomorphic to a subalgebra of A iff $\mathrm{St}(B)$ is a continuous image of $\mathrm{St}(A)$

So $\mathcal{P}(\mathbb{N})/\mathit{fin}$ cannot be embedded into $\mathbb{M}$ and $\mathbb{C}$,
and $\mathbb{M}$ cannot be embedded into $\mathbb{C}$.

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$\mathbb{C}$ into $\mathbb{M}$?

Yes.
A false proof would be to restrict the qutient homomrphism $q : \mathcal{B} \to \mathbb{M}$ to the regular open sets.
Why?
We can have disjoint regular open sets U and V such that $\mu(U \cup V) < 1$ yet $U \cup V$ is dense in $[0, 1]$.
Then $U \vee V = \mathbb{1}$ in $\mathbb{C}$, but $q(U) \vee q(V) < \mathbb{1}$ in $\mathbb{M}$.

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$\mathbb{C}$ into $\mathbb{M}$?

A correct proof is by transfinite recursion, using the completeness of $\mathbb{M}$ to find new values at each step.
Start with the natural inclusion of the algebra generated by the rational intervals.

Sikorski's extension criterion applies at each step.

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Parovičenko's theorem

Theorem

Every Boolean algebra of cardinality $\aleph_1$ (or less) embeds into $\mathcal{P}(\mathbb{N})/\mathit{fin}$.
Hence the Continuum Hypothesis implies that $\mathbb{C}$ and $\mathbb{M}$ can be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$.

Do we need the Continuum Hypothesis?

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$\mathbb{C}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

For $\mathbb{C}$ we don't need CH; let $Q = \mathbb{Q} \cap [0,1]$

- Let $\{V_q : q \in Q\}$ partition $\mathbb{N}$ into infinite sets.
- Define $\Phi(U) = \bigcup\{V_q : q \in Q \cap U\}$ (U regular open).
- $\phi : U \mapsto \Phi(U)/\mathit{fin}$ is an embedding.

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$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

Assume $\phi : \mathbb{M} \to \mathcal{P}(\mathbb{N})/\mathit{fin}$ is an embedding and take a lifting

$$\Phi : \mathcal{B} \to \mathcal{P}(\mathbb{N})$$

That is: $\phi\bigl([B]_{\mathcal{N}}\bigr) = \bigl[\Phi(B)\bigr]_{\mathit{fin}}$ for all B.

Also $\Phi(A \cap B) =^* \Phi(A) \cap \Phi(B)$, etc.
We can ensure equality on the (countable) algebra C generated by the rational intervals.

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$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

For convenience replace $[0,1]$ by $\mathbb{R}$.
Enumerate $\Phi\bigl([n, n+1)\bigr)$ as $\langle k(n,i) : i \in \mathbb{N}\rangle$
Choose $m(n,i) < 3^i$ such that

$$k(n,i) \in \Phi\bigl(n + [m(n,i)3^{-i}, (m(n,i)+1)3^{-i})\bigr)$$

and put

$$U_n = n + \bigcup_{i=1}^{\infty} [m(n,i)3^{-i}, (m(n,i)+1)3^{-i})$$

Observe $\mu(U_n) \le \sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2}$

$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/fin$

Let $U = \bigcup_n U_n$ and $F = \mathbb{R} \setminus U$.

- Then $\mu(F \cap [n, n+1)) \geq \frac{1}{2}$ for all n
- So $\Phi(F) \cap \Phi([n, n+1))$ is infinite.

Take the first i_n with $k(n, i_n) \in \Phi(F)$ and put

$$I_n = n + [m(n, i_n)3^{-i_n}, (m(n, i_n) + 1)3^{-i_n})$$

$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/fin$

- On the one hand $\bigcup_n I_n \cap F = \emptyset$, hence $\Phi(\bigcup_n I_n) \cap \Phi(F) =^* \emptyset$.
- On the other hand: $\bigcup_n \Phi(I_n) \cap \Phi(F) \neq^* \emptyset$
  (it contains $\{k(n, i_n) : n \in \mathbb{N}\}$).

So, if Φ is an embedding of C into $\mathcal{P}(\mathbb{N})$ then there will be sequence like that of the I_n above.

That is, with $\Phi(\bigcup_n I_n) \neq^* \bigcup_n \Phi(I_n)$.

Open Colouring Axiom

OCA

If X is separable and metrizable and $X^2 = K_0 \cup K_1$ with K_0 open and symmetric then

- either there is an uncountable Y such that $Y^2 \subseteq K_0$
- or $X = \bigcup_n X_n$ with $X_n^2 \subseteq K_1$ for all n

This contradicts CH but it is consistent with the usual axioms of Set Theory.

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Open Colouring Axiom

The Open Colouring Axiom implies:

If there is an embedding of $\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/fin$
then there are an embedding ψ and a lifting Ψ such that

- Ψ embeds C into $\mathcal{P}(\mathbb{N})$
- $\bigcup_n \Psi(I_n) =^* \Psi(\bigcup_n I_n)$ for all sequences of intervals as above.

Which means that OCA implies there are no embeddings at all.

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