A tale of three Boolean algebras

Non impeditus ab ulla scientia

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Delft, 13 November, 2008: 16:00–17:00

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Outline

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Borel sets and ideals

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Ideal: closed under finite unions and taking subsets.
$\mathcal{M}$ and $\mathcal{N}$ are even σ-ideals: closed under countable unions.

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Operations: $U \vee V = \operatorname{int}\operatorname{cl}(U \cup V)$, $U \wedge V = U \cap V$ and $U' = [0,1] \setminus \operatorname{cl} U$.

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Unless you know $\mathbb{M}$ very well, in which case ...

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Also $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is not isomorphic to an easily recognizable algebra, unless of course . . .

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The algebras correspond to three ‘familiar’ objects from Fucntional Analysis.

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We’ll see how that works later.

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Tarski's theorem

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This was proved by Tarski in the 1930's.

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Separability, etc

Theorem

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We say $\mathbb{C}$ is σ-centered

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For each n pick B_n in $\mathcal{F}_n$ of measure less that 3^{-n}.
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The union $\bigcup_n B_n$ has measure at most $\frac{1}{2}$; its complement belongs to no $\mathcal{F}_n$.

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For each rational interval $[p, q)$ put

$$\mathcal{L}_{p,q} = \left\{B : \mu\bigl(B \cap [p, q)\bigr) > \frac{1}{2}(q - p)\right\}$$

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$\mathbb{M}$ is the union of the countably many $\mathcal{L}_{p,q}$.

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- if $x \neq y$ then $S_x \cap S_y =^* \emptyset$.
- so $\{S_x : x > 0 \text{ and } x \text{ irrational}\}$ gives the set we seek.

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Stone space

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$\{\bar{a} : a \in B\}$ is a base for a topology on $\mathrm{St}(B)$ that is zero-dimensional compact Hausdorff and has $\{\bar{a} : a \in B\}$ as its family of clopen sets.

Three spaces

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- $\mathrm{St}(\mathcal{P}(\mathbb{N})/\mathit{fin})$ does not satisfy the countable chain condition

Three Banach algebras

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So $\mathcal{P}(\mathbb{N})/\mathit{fin}$ cannot be embedded into $\mathbb{M}$ and $\mathbb{C}$,
and $\mathbb{M}$ cannot be embedded into $\mathbb{C}$.

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Why?

We can have disjoint regular open sets U and V such that $\mu(U \cup V) < 1$ yet $U \cup V$ is dense in $[0, 1]$.

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Why?

We can have disjoint regular open sets U and V such that $\mu(U \cup V) < 1$ yet $U \cup V$ is dense in $[0, 1]$.

Then $U \vee V = \mathbb{1}$ in $\mathbb{C}$, but $q(U) \vee q(V) < \mathbb{1}$ in $\mathbb{M}$.

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Start with the natural inclusion of the algebra generated by the rational intervals.

Sikorski's extension criterion applies at each step.

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Parovičenko's theorem

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Do we need the Continuum Hypothesis?

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- Define $\Phi(U) = \bigcup\{V_q : q \in Q \cap U\}$ (U regular open).
- $\phi : U \mapsto \Phi(U)/\mathit{fin}$ is an embedding.

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$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

Assume $\phi : \mathbb{M} \to \mathcal{P}(\mathbb{N})/\mathit{fin}$ is an embedding and take a lifting

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Also $\Phi(A \cap B) =^* \Phi(A) \cap \Phi(B)$, etc.

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Also $\Phi(A \cap B) =^* \Phi(A) \cap \Phi(B)$, etc.
We can ensure equality on the (countable) algebra C generated by the rational intervals.

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$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

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Enumerate $\Phi\bigl([n,n+1)\bigr)$ as $\langle k(n,i) : i \in \mathbb{N}\rangle$
Choose $m(n,i) < 3^i$ such that

$$k(n,i) \in \Phi\bigl(n + [m(n,i)3^{-i}, (m(n,i)+1)3^{-i})\bigr)$$

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and put

$$U_n = n + \bigcup_{i=1}^{\infty}[m(n,i)3^{-i}, (m(n,i)+1)3^{-i})$$

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$$U_n = n + \bigcup_{i=1}^{\infty} [m(n,i)3^{-i}, (m(n,i)+1)3^{-i})$$

Observe $\mu(U_n) \le \sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2}$

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$\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

Let $U = \bigcup_n U_n$ and $F = \mathbb{R} \setminus U$.

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Let $U = \bigcup_n U_n$ and $F = \mathbb{R} \setminus U$.

- Then $\mu(F \cap [n, n+1)) \geq \frac{1}{2}$ for all n

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That is, with $\Phi(\bigcup_n I_n) \neq^* \bigcup_n \Phi(I_n)$.

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This contradicts CH but it is consistent with the usual axioms of Set Theory.

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Which means that OCA implies there are no embeddings at all.

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