A concrete co-existential map that is not confluent

Conspici Quam Prodesse

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Outline

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Two maps

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- $\{\alpha\} \times X$ for $\alpha \in \kappa$ (that's just X), and
- $X_u = \beta\pi_\kappa^{\leftarrow}(u)$ for $u \in \kappa^*$ (an enriched version of X).

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Also, $2^{\kappa\times X}$ is naturally isomorphic to $(2^X)^\kappa$:
$\langle A_\alpha : \alpha < \kappa\rangle$ corresponds to $\bigcup_\alpha \{\alpha\} \times A_\alpha$.

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The in/prefix 'co' is there because π_u and e_u work in opposite directions.

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Of course niceness is in the mind of the considerer.

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A continuous surjection $f : Y \to X$ is *co-existential* if there are an ultrafilter u on some κ and a continuous surjection $g : X_u \to Y$ such that $f \circ g = \pi_u$.

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The corresponding embedding $(A \mapsto f^{\leftarrow}[A])$ of 2^X into 2^Y is an *existential* embedding.

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confluent: if for every continuum C in X *every* component of $f^{\leftarrow}[C]$ maps onto C

weakly confluent: if for every continuum C in X *some* component of $f^{\leftarrow}[C]$ maps onto C

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If $\pi_u = f \circ g$ then the component of $g[C_u]$ in $f^{\leftarrow}[C]$ maps onto C.

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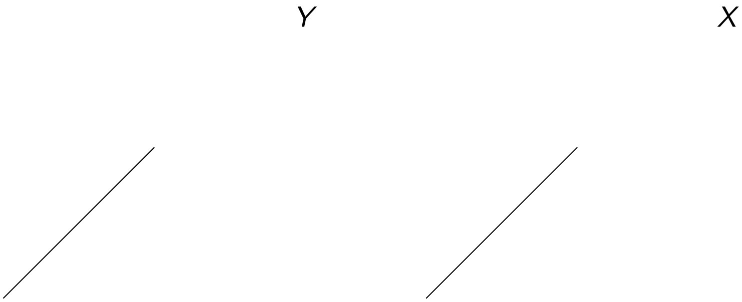
But one would be wrong.

An example



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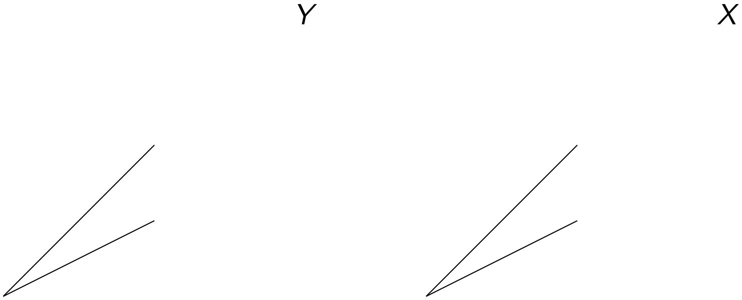

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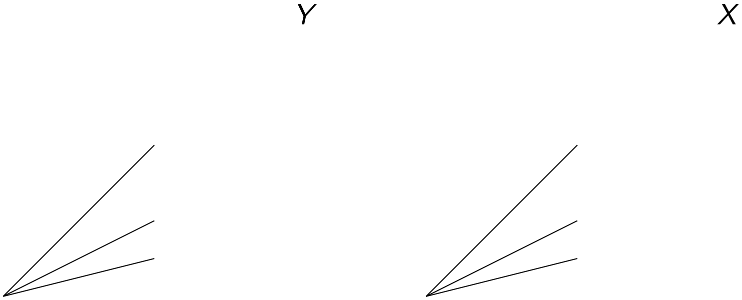

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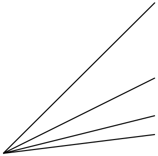

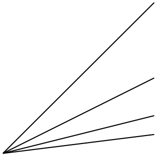

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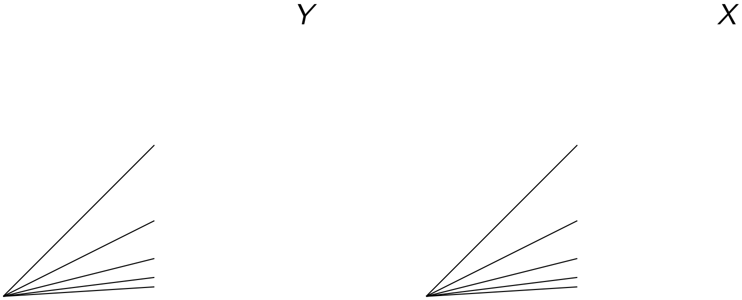

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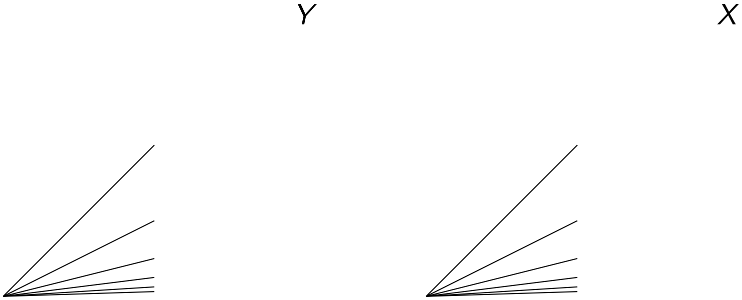

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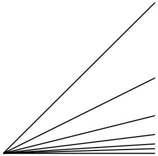

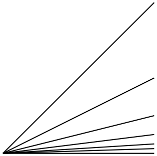

The infinite broom B

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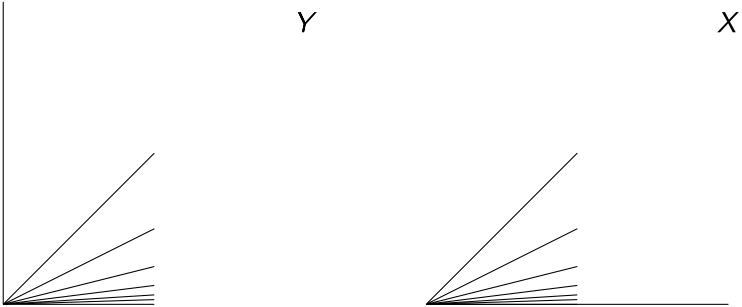

The infinite broom B plus an extra hair

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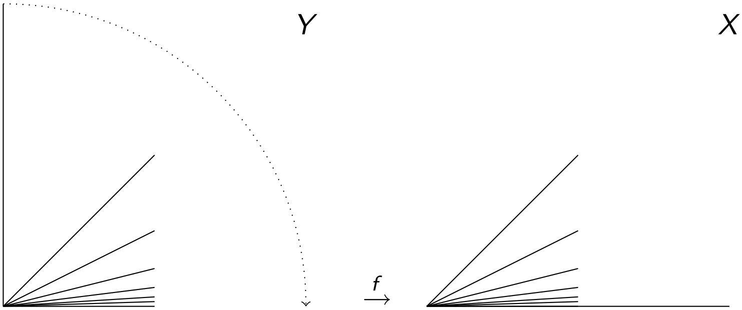

The infinite broom B plus an extra hair and the map f.

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Codomain: $X = B \cup ([0,2] \times \{0\})$.

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Map f: identity on B and $f(0,y) = \langle y, 0\rangle$.

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The component $\bigl\{\langle 1,0\rangle\bigr\}$ does not map onto C.

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We can define a map $g : \omega \times X \to Y$ such that $f \circ g_u = \pi_u$ for all $u \in \omega^*$; where $g_u = \beta g \restriction X_u$.

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$$g(n, x, y) = \begin{cases} \langle x, y \rangle & \text{if } \langle x, y \rangle \in \bigcup_{i \leqslant n} H_i \\ \langle 0, x \rangle & \text{if } \langle x, y \rangle \in \bigcup_{i > n} H_i \cup \big(\{0\} \times [0, 2]\big) \end{cases}$$

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For each $u \in \omega^*$ we get $g_u[F_u] = B$ and $g_u[G_u] = \{0\} \times [0, 2]$; so g_u is always onto.

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It is $[n, \omega)$ if $\langle x, y\rangle \in H_n$ and it is ω if $y = 0$.

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$\pi_u^{\leftarrow}[C]$ splits into $g_u^{\leftarrow}\big[\{0\} \times [1,2]\big]$ and $g_u^{\leftarrow}(1,0)$.

The latter set even maps onto ω^* so it has plenty components that do not map onto C under π_u.

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one wonders whether these maps are as nice topologically as their duals are algebraically.

This merits further investigation.

Outline

Light reading

Website: `fa.its.tudelft.nl/~hart`

P. Bankston,
Not every co-existential map is confluent, Houston Journal of Mathematics, to appear.

K. P. Hart.
A concrete co-existential map that is not confluent, Topology Proceedings, to appear.