Lelek's problem is not a metric problem Conspici Quam Prodesse

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- 2 The Problem
- 3 The conversion
- A better reflection







Definition

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[0,1] is chainable; the circle S^1 is not.





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A continuum, X, has xxx span zero if every subcontinuum Z of $X \times X$ that satisfies yyy intersects the diagonal $\{\langle x, x \rangle : x \in X\}$.





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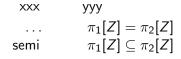
 $\begin{array}{ll} \mathsf{xxx} & \mathsf{yyy} \\ \ldots & \pi_1[Z] = \pi_2[Z] \end{array}$





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xxx ... semi surjective

yyy

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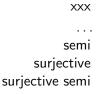


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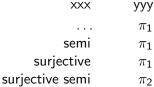
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[0,1] has all spans zero, S^1 has all spans non-zero

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Question (Lelek)

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What about the converse?

This is an important problem in metric continuum theory. We free it from the metric constraints.



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A useful tool

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Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L. wL is the Wallman space of L.

Many properties of a space X are first-order when expressed in terms of 2^X , its lattice of (all) closed sets.

Quite often, in the case of wL, it suffices to work in L only.



Reflection

Theorem

Any counterexample to Lelek's problem can be converted into a metrizable counterexample.



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Let X be a counterexample, let $L \prec 2^X$ (an elementary sublattice).



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Not quite ...

Complications

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$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)$$
$$((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \rightarrow \bigvee_{n \in \omega} \Phi_n(u_1, u_2, u_3, u_4))$$



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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an *n*-element chain refinement.

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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an *n*-element chain refinement.

It suffices to consider four-element open covers only.

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Another complication

We have no decent formula, $L_{\omega_{1},\omega}$ or otherwise, that describes in terms of 2^{X} that X has span (non-)zero.







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Theorem

In this situation:

- wL is chainable iff X is chainable
- wL has span zero iff X has span zero (any kind)



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Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.



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Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.

Furthermore, one needs only consider covers and refinements that belong to a certain base.





Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$.



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For the converse ...



Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.



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Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK} .



Span zero, the real argument

Apply Shelah's Ultrapower theorem



Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal κ , an ultrafilter u on κ and an isomorphism $h: \prod_u (2^{X \times X}) \to \prod_u wK$ (which can be taken to be the identity on K).



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How does that help?

For that we need some topology.



Dualizing ultrapowers

Take a compact Hausdorff space Y with a lattice base B. Also take a cardinal κ and an ultrafilter u on κ .



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The Wallman space of the ultrapower $\prod_{u} B$ is the fiber $p_{\kappa}^{\leftarrow}(u)$. Bankston calls this the ultracopower of Y; we write Y_{u} .

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Back to
$$Z \subseteq wK$$
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Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.



Comment from Piotr Minc

Lelek's problem *is* a metric problem.



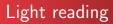




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D. Bartošová, K. P. Hart, B. van der Steeg,

Lelek's problem is not a metric problem, to appear.

