Model theory is useful in Topology

Conspici Quam Prodesse

K. P. Hart

Math & Stat
Miami University

Madison, 7 April, 2009: 16:30–17:30

Outline

Outline

Lattice to space

Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L.

Lattice to space

Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L. wL is the Wallman space of L.

Lattice to space

Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L. wL is the Wallman space of L.

- wL is the set of ultrafilters on L

Lattice to space

Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L. wL is the Wallman space of L.

- wL is the set of ultrafilters on L
- $\{\bar{a} : a \in L\}$ is the base for the closed sets

Lattice to space

Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L. wL is the Wallman space of L.

- wL is the set of ultrafilters on L
- $\{\bar{a} : a \in L\}$ is the base for the closed sets
- where $\bar{a} = \{u \in wL : a \in u\}$

Space to lattice

A compact Hausdorff space X corresponds to many lattices:

Space to lattice

A compact Hausdorff space X corresponds to many lattices:

- the family 2^X of all closed subsets of X

Space to lattice

A compact Hausdorff space X corresponds to many lattices:

- the family 2^X of all closed subsets of X
- any base for the closed sets that is closed under finite unions and intersections; a *lattice-base*

Space to lattice

A compact Hausdorff space X corresponds to many lattices:

- the family 2^X of all closed subsets of X
- any base for the closed sets that is closed under finite unions and intersections; a *lattice-base*

Many properties of X are first-order properties of 2^X or, better still, of all lattice-bases.

Outline

The theorem

Theorem

Every metric continuum, X, is the weakly confluent image of a one-dimensional hereditarily indecomposable continuum.

The theorem

Theorem

Every metric continuum, X, is the weakly confluent image of a one-dimensional hereditarily indecomposable continuum.

- weakly confluent: each subconinuum of X is the image of some subcontinuum of the domain

The theorem

Theorem

Every metric continuum, X, is the weakly confluent image of a one-dimensional hereditarily indecomposable continuum.

- weakly confluent: each subconinuum of X is the image of some subcontinuum of the domain
- one-dimensional: covering dimension is one

The theorem

Theorem

Every metric continuum, X, is the weakly confluent image of a one-dimensional hereditarily indecomposable continuum.

- weakly confluent: each subconinuum of X is the image of some subcontinuum of the domain
- one-dimensional: covering dimension is one
- hereditarily indecomposable: if two subcontinua meet then they are comparable (with respect to $\subseteq$)

Formulas

- a is connected: conn(a) abbreviates

$$(\forall x)(\forall y)\Big[\big((x \cap y = 0) \wedge (x \cup y = a)\big) \implies \big((x = 0) \vee (x = a)\big)\Big],$$

Formulas

- a is connected: conn(a) abbreviates

$$(\forall x)(\forall y)\Big[\big((x \cap y = 0) \wedge (x \cup y = a)\big) \implies \big((x = 0) \vee (x = a)\big)\Big],$$

- one-dimensonal: connected plus

$$(\forall x_1)(\forall x_2)(\forall x_3)(\exists y_1)(\exists y_2)(\exists y_3)$$
$$\Big[(x_1 \cap x_2 \cap x_3 = 0) \implies \big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge (x_3 \leqslant y_3) \wedge$$
$$\wedge (y_1 \cap y_2 \cap y_3 = 0) \wedge (y_1 \cup y_2 \cup y_3 = 1)\big)\Big]$$

One more formula

Hereditary indecomposability can be expressed thus:

One more formula

Hereditary indecomposability can be expressed thus:

$$
\begin{aligned}
&(\forall x)(\forall y)(\forall u)(\forall v)(\exists z_1)(\exists z_2)(\exists z_3)\\
&\qquad\Big[\big((x \cap y = 0) \wedge (x \cap u = 0) \wedge (y \cap v = 0)\big) \implies\\
&\implies \big((x \cap (z_2 \cup z_3) = 0) \wedge (y \cap (z_1 \cup z_2) = 0) \wedge (z_1 \cap z_3 = 0)\\
&\wedge (z_1 \cap z_2 \cap v = 0) \wedge (z_2 \cap z_3 \cap u = 0) \wedge (z_1 \cup z_2 \cup z_3 = 1)\big)\Big].
\end{aligned}
$$

One more formula

Hereditary indecomposability can be expressed thus:

$$
\begin{aligned}
&(\forall x)(\forall y)(\forall u)(\forall v)(\exists z_1)(\exists z_2)(\exists z_3) \\
&\qquad \Big[\big((x \cap y = 0) \wedge (x \cap u = 0) \wedge (y \cap v = 0)\big) \implies \\
&\implies \big((x \cap (z_2 \cup z_3) = 0) \wedge (y \cap (z_1 \cup z_2) = 0) \wedge (z_1 \cap z_3 = 0) \\
&\wedge (z_1 \cap z_2 \cap v = 0) \wedge (z_2 \cap z_3 \cap u = 0) \wedge (z_1 \cup z_2 \cup z_3 = 1)\big)\Big].
\end{aligned}
$$

This was found by Krasinkiewicz and Minc.

A self-generalizing theorem

Theorem

Any continuum, X, is the weakly confluent image of a one-dimensional hereditarily indecomposable continuum of the same weight as X.

A self-generalizing theorem

Theorem

Any continuum, X, is the weakly confluent image of a one-dimensional hereditarily indecomposable continuum of the same weight as X.

Proof.

We build a specific theory in the language of lattices, prove it consistent and take a model. The Wallman space of the model is the desired preimage, because we ensure the model contains an isomorphic copy of the lattice 2^X. □

The theory

First we introduce constants: all elements of 2^X

The theory

First we introduce constants: all elements of 2^X and for each connected element, A, of 2^X a copy A^*.

The theory

First we introduce constants: all elements of 2^X and for each connected element, A, of 2^X a copy A^*.

The theory consists of

The theory

First we introduce constants: all elements of 2^X and for each connected element, A, of 2^X a copy A^*.

The theory consists of

- The axioms for a distributive lattice.

The theory

First we introduce constants: all elements of 2^X and for each connected element, A, of 2^X a copy A^*.
The theory consists of

- The axioms for a distributive lattice.
- Formulas that express separativity, normality, one-dimensionality, connectivity and hereditary indecomposability.

The theory

First we introduce constants: all elements of 2^X and for each connected element, A, of 2^X a copy A^*.
The theory consists of

- The axioms for a distributive lattice.
- Formulas that express separativity, normality, one-dimensionality, connectivity and hereditary indecomposability.
- The diagram of 2^X, i.e., the multiplication tables of $\cap$ and $\cup$.

The theory

First we introduce constants: all elements of 2^X and for each connected element, A, of 2^X a copy A^*.
The theory consists of

- The axioms for a distributive lattice.
- Formulas that express separativity, normality, one-dimensionality, connectivity and hereditary indecomposability.
- The diagram of 2^X, i.e., the multiplication tables of $\cap$ and $\cup$.

This ensures that the Wallman space of the resulting lattice has the desired properties and maps onto X.

The theory

We also need

The theory

We also need

- $\mathrm{conn}(A^*)$ for every connected $A \in 2^X$.

The theory

We also need

- $\mathrm{conn}(A^*)$ for every connected $A \in 2^X$.
- $A^* \subseteq A$ for every connected $A \in 2^X$.

The theory

We also need

- $\mathrm{conn}(A^*)$ for every connected $A \in 2^X$.
- $A^* \subseteq A$ for every connected $A \in 2^X$.
- $(A^* \subseteq B) \to (A \subseteq B)$
  for every connected $A \in 2^X$ and all $B \in 2^X$.

The theory

We also need

- $\mathrm{conn}(A^*)$ for every connected $A \in 2^X$.
- $A^* \subseteq A$ for every connected $A \in 2^X$.
- $(A^* \subseteq B) \rightarrow (A \subseteq B)$
  for every connected $A \in 2^X$ and all $B \in 2^X$.

Then (the interpretation of) A^* will be a subcontinuum of the Wallman space that maps exactly onto A.

Finishing the proof

- The Maćkowiak-Tymchatyn theorem implies that every *countable* subset of our theory has a model.

Finishing the proof

- The Maćkowiak-Tymchatyn theorem implies that every *countable* subset of our theory has a model.
- Hence the whole theory has a model . . . done.

Finishing the proof

- The Maćkowiak-Tymchatyn theorem implies that every *countable* subset of our theory has a model.
- Hence the whole theory has a model . . . done.
- Well, almost: Löwenheim-Skolem helps get the weight down to that of X.

A new proof

Interesting by-product.

A new proof

Interesting by-product.
The standard proof of the completeness theorem — adding witnesses — can in this case be converted into an inverse-limit proof of the original metric result.

Outline

The result

Given a metric continuum X there is another metric continuum Y such that

The result

Given a metric continuum X there is another metric continuum Y such that

- X and Y look the same
  (they have elementarily equivalent countable bases)

The result

Given a metric continuum X there is another metric continuum Y such that

- X and Y look the same
  (they have elementarily equivalent countable bases)
- X and Y are not homeomorphic

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\big((x \sqcap y = 0) \to ((x \leqslant u) \wedge (y \leqslant v) \wedge (u \sqcap v = 0) \wedge (u \sqcup v = 1))\big)$$

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\big((x \sqcap y = 0) \rightarrow ((x \leqslant u) \wedge (y \leqslant v) \wedge (u \sqcap v = 0) \wedge (u \sqcup v = 1))\big)$$

In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\big((x \sqcap y = 0) \to ((x \leqslant u) \wedge (y \leqslant v) \wedge (u \sqcap v = 0) \wedge (u \sqcup v = 1))\big)$$

In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).
By *compactness*, if some base satisfies this sentence then the space is zero-dimensional.

Example: no isolated points

Here is a another first-order sentence, call it π

$$(\forall x)(\exists y)\big((x < 1) \rightarrow ((x < y) \land (y < 1))\big)$$

Example: no isolated points

Here is a another first-order sentence, call it π

$$(\forall x)(\exists y)\big((x < 1) \to ((x < y) \wedge (y < 1))\big)$$

In words: every closed proper subset (x) is properly contained in a closed proper subset (y);

Example: no isolated points

Here is a another first-order sentence, call it π

$$(\forall x)(\exists y)\big((x < 1) \to ((x < y) \wedge (y < 1))\big)$$

In words: every closed proper subset (x) is properly contained in a closed proper subset (y);
in fewer words: there are no isolated points.

Example: no isolated points

Here is a another first-order sentence, call it π

$$(\forall x)(\exists y)\bigl((x < 1) \to ((x < y) \wedge (y < 1))\bigr)$$

In words: every closed proper subset (x) is properly contained in a closed proper subset (y);
in fewer words: there are no isolated points.
If some base satisfies this sentence then the space has no isolated points.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.
Then X is zero-dimensional and without isolated points.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.
Then X is zero-dimensional and without isolated points.
So X is (homeomorphic to) the Cantor set C.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.
Then X is zero-dimensional and without isolated points.
So X is (homeomorphic to) the Cantor set C.

Thus: if X looks like C then X is homeomorphic to C.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.
Then X is zero-dimensional and without isolated points.
So X is (homeomorphic to) the Cantor set C.

Thus: if X looks like C then X is homeomorphic to C.

The Cantor set is categorical among compact metric spaces.

What the main result says

Among metric continua there is no categorical space.

What the main result says

Among metric continua there is no categorical space.
No (in)finite list of first-order properties will characterize a single metric continuum.

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and
- chainable

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and
- chainable

A two-item list but . . .

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and
- chainable

A two-item list but . . .
Chainability is *not* first-order.

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and
- chainable

A two-item list but ...
Chainability is *not* first-order.
(Hereditary indecomposability is.)

An embedding lemma

Lemma

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
There is an embedding of $\mathcal{C}$ into the ultrapower of $\mathcal{B}$ by u.

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
Let $\varphi : \mathcal{C} \to \mathcal{B}_u$ be an embedding.

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
Let $\varphi : \mathcal{C} \to \mathcal{B}_u$ be an embedding.

Apply the Löwenheim-Skolem theorem:

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
Let $\varphi : \mathcal{C} \to \mathcal{B}_u$ be an embedding.

Apply the Löwenheim-Skolem theorem:
Find a countable elementary sublattice $\mathcal{D}$ of $\mathcal{B}_u$ that contains $\varphi[\mathcal{C}]$.

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
Let $\varphi : \mathcal{C} \to \mathcal{B}_u$ be an embedding.

Apply the Löwenheim-Skolem theorem:
Find a countable elementary sublattice $\mathcal{D}$ of $\mathcal{B}_u$ that contains $\varphi[\mathcal{C}]$.
Let Y be the Wallman space of $\mathcal{D}$.

Properties of Y

- Y is compact metric ($\mathcal{D}$ is countable).

Properties of Y

- Y is compact metric ($\mathcal{D}$ is countable).
- $\mathcal{D}$ is a base for the closed sets of Y (by Wallman's theorem).

Properties of Y

- Y is compact metric ($\mathcal{D}$ is countable).
- $\mathcal{D}$ is a base for the closed sets of Y (by Wallman's theorem).
- $\mathcal{D}$ is elementarily equivalent to $\mathcal{B}_u$ and hence to $\mathcal{B}$.

Properties of Y

- Y is compact metric ($\mathcal{D}$ is countable).
- $\mathcal{D}$ is a base for the closed sets of Y (by Wallman's theorem).
- $\mathcal{D}$ is elementarily equivalent to $\mathcal{B}_u$ and hence to $\mathcal{B}$.
- Y maps onto Z (because $\varphi[\mathcal{C}]$ is embedded into $\mathcal{D}$).

Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets.

Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets. There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).

Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets.
There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).
Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and

Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets.
There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).
Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and such that Y maps onto Z.

Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets.
There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).
Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and such that Y maps onto Z.
So: Y is not homeomorphic to X.

Outline

Covering dimension

Definition

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$

Covering dimension

Definition

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

Covering dimension

Definition

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

There is a convenient first-order characterization.

Covering dimension

Definition

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

There is a convenient first-order characterization.

Theorem (Hemmingsen)

$\dim X \leqslant n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

Large inductive dimension

Definition

$\operatorname{Ind} X \leqslant n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\operatorname{Ind} L \leqslant n-1$.

Large inductive dimension

Definition

$\operatorname{Ind} X \leqslant n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\operatorname{Ind} L \leqslant n-1$.
The starting point: $\operatorname{Ind} X \leqslant -1$ iff $X = \emptyset$.

Large inductive dimension

Definition

$\operatorname{Ind} X \leqslant n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\operatorname{Ind} L \leqslant n-1$.
The starting point: $\operatorname{Ind} X \leqslant -1$ iff $X = \emptyset$.

L is a partition between A and B means:

Large inductive dimension

Definition

$\operatorname{Ind} X \leqslant n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\operatorname{Ind} L \leqslant n-1$.
The starting point: $\operatorname{Ind} X \leqslant -1$ iff $X = \emptyset$.

L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

Dimensionsgrad

Definition

$\operatorname{Dg} X \leqslant n$ between every two disjoint closed sets A and B there is a cut C that satisfies $\operatorname{Dg} C \leqslant n-1$.

Dimensionsgrad

Definition

$\operatorname{Dg} X \leqslant n$ between every two disjoint closed sets A and B there is a cut C that satisfies $\operatorname{Dg} C \leqslant n-1$.

The starting point: $\operatorname{Dg} X \leqslant -1$ iff $X = \emptyset$.

Dimensionsgrad

Definition

$\operatorname{Dg} X \leqslant n$ between every two disjoint closed sets A and B there is a cut C that satisfies $\operatorname{Dg} C \leqslant n-1$.
The starting point: $\operatorname{Dg} X \leqslant -1$ iff $X = \emptyset$.

C is a cut between A and B means:

Dimensionsgrad

Definition

$\mathrm{Dg}\,X \leqslant n$ between every two disjoint closed sets A and B there is a cut C that satisfies $\mathrm{Dg}\,C \leqslant n-1$.
The starting point: $\mathrm{Dg}\,X \leqslant -1$ iff $X = \emptyset$.

C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

(In)equalities

- For σ-compact metric X: $\dim X$

(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X$

(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X = \operatorname{Dg} X$

(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X = \operatorname{Dg} X$
- The first equality is classical and holds for all metric X

(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X = \operatorname{Dg} X$
- The first equality is classical and holds for all metric X
- the second is fairly recent (1999).

(In)equalities

- For σ-compact metric X: $\dim X = \operatorname{Ind} X = \operatorname{Dg} X$
- The first equality is classical and holds for all metric X
- the second is fairly recent (1999).
- There is for each n a locally connected Polish X_n with $\operatorname{Dg} X_n = 1$ and $\dim X_n = n$ (Fedorchuk, van Mill)

More inequalities

For arbitrary compact Hausdorff spaces

More inequalities

For arbitrary compact Hausdorff spaces:

- $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ (each partition is a cut)

More inequalities

For arbitrary compact Hausdorff spaces:

- $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ (each partition is a cut)
- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)

More inequalities

For arbitrary compact Hausdorff spaces:

- $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ (each partition is a cut)
- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)
- $\dim X \leqslant \operatorname{Dg} X$ (Fedorchuk)

More inequalities

For arbitrary compact Hausdorff spaces:

- $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ (each partition is a cut)
- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)
- $\dim X \leqslant \operatorname{Dg} X$ (Fedorchuk)

We will reprove the last two inequalities.

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets and cast as a formula, δ_n, in the language of lattices

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets and cast as a formula, δ_n, in the language of lattices

$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2}) \\
&\qquad \big[(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = 0) \implies \\
&\qquad ((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \cdots \wedge (x_{n+2} \leqslant y_{n+2}) \\
&\qquad \wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = 0) \\
&\qquad \wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1))\big].
\end{aligned}
$$

Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$(\forall x)(\forall y)(\exists u)$$
$$\Big[\Big(\big((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\big) \implies \big(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\big)\Big]$$

Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$(\forall x)(\forall y)(\exists u)$$
$$\Bigl[\Bigl(\bigl((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\bigr) \implies \bigl(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\bigr)\Bigr]$$

where $\operatorname{partn}(u, x, y, a)$ says that u is a partition between x and y in the (sub)space a:

Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$(\forall x)(\forall y)(\exists u)$$
$$\Big[\Big(\big((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\big) \implies \big(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\big)\Big)\Big]$$

where $\operatorname{partn}(u, x, y, a)$ says that u is a partition between x and y in the (sub)space a:

$$(\exists f)(\exists g)\big((x \cap f = 0) \wedge (y \cap g = 0) \wedge (f \cup g = a) \wedge (f \cap g = u)\big).$$

Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$(\forall x)(\forall y)(\exists u)$$
$$\Big[\big(\big((x \leqslant a)\wedge(y \leqslant a)\wedge(x\cap y = \mathsf{o})\big) \implies \big(\operatorname{partn}(u,x,y,a)\wedge I_{n-1}(u)\big)\Big]$$

where $\operatorname{partn}(u,x,y,a)$ says that u is a partition between x and y in the (sub)space a:

$$(\exists f)(\exists g)\big((x \cap f = \mathsf{o}) \wedge (y \cap g = \mathsf{o}) \wedge (f \cup g = a) \wedge (f \cap g = u)\big).$$

We start with $I_{-1}(a)$, which denotes $a = \mathsf{o}$

Dimensionsgrad

Here we have the recursive definition of a formula $\Delta_n(a)$:

Dimensionsgrad

Here we have the recursive definition of a formula $\Delta_n(a)$:

$$(\forall x)(\forall y)(\exists u)$$
$$\Bigl[\bigl((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\bigr) \implies \bigl(\mathrm{cut}(u, x, y, a) \wedge \Delta_{n-1}(u)\bigr)\Bigr],$$

Dimensionsgrad

Here we have the recursive definition of a formula $\Delta_n(a)$:

$$(\forall x)(\forall y)(\exists u)$$
$$\Bigl[\bigl((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\bigr) \implies \bigl(\mathrm{cut}(u, x, y, a) \wedge \Delta_{n-1}(u)\bigr)\Bigr],$$

and $\Delta_{-1}(a)$ denotes $a = 0$.

Dimensionsgrad (auxiliary formulas)

The formula $\mathrm{cut}(u, x, y, a)$ expresses that u is a cut between x and y in a:

Dimensionsgrad (auxiliary formulas)

The formula $\mathrm{cut}(u, x, y, a)$ expresses that u is a cut between x and y in a:

$$(\forall v)\big[\big((v \leqslant a) \wedge \mathrm{conn}(v) \wedge (v \sqcap x \neq 0) \wedge (v \sqcap y \neq 0)\big) \implies (v \sqcap u \neq 0)\big],$$

Dimensionsgrad (auxiliary formulas)

The formula $\mathrm{cut}(u, x, y, a)$ expresses that u is a cut between x and y in a:

$$(\forall v)\big[((v \leqslant a) \wedge \mathrm{conn}(v) \wedge (v \cap x \neq 0) \wedge (v \cap y \neq 0)) \implies (v \cap u \neq 0)\big],$$

and $\mathrm{conn}(a)$ says that a is connected:

Dimensionsgrad (auxiliary formulas)

The formula $\mathrm{cut}(u,x,y,a)$ expresses that u is a cut between x and y in a:

$$(\forall v)\big[((v \leqslant a) \wedge \mathrm{conn}(v) \wedge (v \cap x \neq \mathsf{o}) \wedge (v \cap y \neq \mathsf{o})) \implies (v \cap u \neq \mathsf{o})\big],$$

and $\mathrm{conn}(a)$ says that a is connected:

$$(\forall x)(\forall y)\big[((x \cap y = \mathsf{o}) \wedge (x \cup y = a)) \implies ((x = \mathsf{o}) \vee (x = a))\big],$$

Why formulas?

- $\dim X \leqslant n$ iff $2^X \vDash \delta_n$

Why formulas?

- $\dim X \leqslant n$ iff $2^X \vDash \delta_n$
- $\operatorname{Ind} X \leqslant n$ iff $2^X \vDash I_n(X)$

Why formulas?

- $\dim X \leqslant n$ iff $2^X \vDash \delta_n$
- $\operatorname{Ind} X \leqslant n$ iff $2^X \vDash I_n(X)$
- $\operatorname{Dg} X \leqslant n$ iff $2^X \vDash \Delta_n(X)$

Covering dimension

Theorem

Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Covering dimension

Theorem

Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Proof.

Both directions use swelling and shrinking to replace the finite families by combinatorially equivalent subfamilies of the base. □

Large inductive dimension

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

Large inductive dimension

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

Proof.

Induction: given A and B expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between A' and B', such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As $\mathcal{B}_L$ is a base for the closed sets of L we know, by inductive assumption, that $\operatorname{Ind} L \leqslant n - 1$. □

Large inductive dimension

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

Proof.

Induction: given A and B expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between A' and B', such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As $\mathcal{B}_L$ is a base for the closed sets of L we know, by inductive assumption, that $\operatorname{Ind} L \leqslant n - 1$. □

No equivalence, see later.

Dimensionsgrad

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then

Dimensionsgrad

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Dimensionsgrad

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Proof.

Let $X = [0,1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0,q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p,1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).

Dimensionsgrad

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Proof.

Let $X = [0,1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0,q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p,1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).
$\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously

Dimensionsgrad

Theorem

Let X be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Proof.

Let $X = [0,1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0,q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p,1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).
$\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously but $\operatorname{Dg}[0,1] = 1$. □

Covering dimension

Theorem

$\dim wL = \dim X$

Covering dimension

Theorem

$\dim wL = \dim X$

Proof.

By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$.

Covering dimension

Theorem

$\dim wL = \dim X$

Proof.

By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does.

Covering dimension

Theorem

$\dim wL = \dim X$

Proof.

By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does.
It follows that $\dim X \leqslant n$ iff $\dim wL \leqslant n$ for all n. □

Large inductive dimension

Theorem

$\operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Large inductive dimension

Theorem

$\operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Proof.

By elementarity we see that $2^X \vDash I_n(X)$ iff $L \vDash I_n(X)$.

Large inductive dimension

Theorem

$\operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Proof.

By elementarity we see that $2^X \models I_n(X)$ iff $L \models I_n(X)$. By previous theorem we know $\operatorname{Ind} wL \leqslant n$, whenever L satisfies $I_n(wL)$.

Large inductive dimension

Theorem

$\operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Proof.

By elementarity we see that $2^X \vDash I_n(X)$ iff $L \vDash I_n(X)$. By previous theorem we know $\operatorname{Ind} wL \leqslant n$, whenever L satisfies $I_n(wL)$. Thus: $\operatorname{Ind} X \leqslant n$ implies $\operatorname{Ind} wL \leqslant n$. $\square$

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Nonproof

By elementarity we see that $2^X \vDash \Delta_n(X)$ iff $L \vDash \Delta_n(X)$.

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Nonproof

By elementarity we see that $2^X \vDash \Delta_n(X)$ iff $L \vDash \Delta_n(X)$.
By previous theorem we know nothing yet about $\operatorname{Dg} wL$.

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n-1$.

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n-1$.
Inductive assumption: $\operatorname{Dg} C \leqslant n-1$ in wL

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n-1$.
Inductive assumption: $\operatorname{Dg} C \leqslant n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.

Dimensionsgrad

Theorem

$\operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Proof.

Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n-1$.
Inductive assumption: $\operatorname{Dg} C \leqslant n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.
Still to show: C-in-wL is a cut between A and B in wL. □

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

Proof (continued)

Let F be a closed set in wL that meets A and B but not C.
We show F is not connected.
Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.
Down in wL we have exactly the same relations, so H_A and H_B show F is not connected.

Finishing up

Let X be compact Hausdorff and let L be a *countable* elementary sublattice of 2^X. Then

Finishing up

Let X be compact Hausdorff and let L be a *countable* elementary sublattice of 2^X. Then

Vedenissof: $\dim X = \dim wL = \operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Finishing up

Let X be compact Hausdorff and let L be a *countable* elementary sublattice of 2^X. Then

Vedenissof: $\dim X = \dim wL = \operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Fedorchuk: $\dim X = \dim wL = \operatorname{Dg} wL \leqslant \operatorname{Dg} X$

Finishing up

Let X be compact Hausdorff and let L be a *countable* elementary sublattice of 2^X. Then

Vedenissof: $\dim X = \dim wL = \operatorname{Ind} wL \leqslant \operatorname{Ind} X$

Fedorchuk: $\dim X = \dim wL = \operatorname{Dg} wL \leqslant \operatorname{Dg} X$

There are X with $\dim X < \operatorname{Dg} X$, so $\operatorname{Dg} wL < \operatorname{Dg} X$ and $\operatorname{Ind} wL < \operatorname{Ind} X$ are possible.

Outline

Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart, J. van Mill and R. Pol.
Remarks on hereditarily indecomposable continua, Topology Proceedings, **25** (2000), 179–206.

K. P. Hart and B. J. van der Steeg.
On the Mackowiak-Tymchatyn theorem, Acta Universitatis Carolinae—Mathematica et Physica, **43** (2002), 27–43.

Light reading

K. P. Hart.
There is no categorical metric continuum, to appear.

K. P. Hart.
Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.