Elementarity my dear Watson

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Outline

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The theorem

Just so we know what we are talking about.

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We consider two languages:

- lattice theory
- set theory

Why lattices?

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- We can use wL to get information about X.
- Or reduce general problems to the metrizable case.

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- The map $q_L : x \mapsto \{a \in L : x \in a\}$ is a continuous surjection.
- Because L is countable, the space wL is metrizable

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Theorem (Hemmingsen)

$\dim X \leq n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

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L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

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C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

(In)equalities

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- the second is fairly recent (1999).
- There is for each n a locally connected Polish X_n with $\operatorname{Dg} X_n = 1$ and $\dim X_n = n$ (Fedorchuk, van Mill)

More inequalities

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- $\dim X \leq \operatorname{Ind} X$ (Vedenissof)
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We will reprove the last two inequalities.

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$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2}) \\
&\qquad \bigl[(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = 0) \implies \\
&\qquad \bigl((x_1 \le y_1) \wedge (x_2 \le y_2) \wedge \cdots \wedge (x_{n+2} \le y_{n+2}) \\
&\qquad \wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = 0) \\
&\qquad \wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1)\bigr)\bigr].
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$$(\forall x)(\forall y)(\exists u)$$
$$\Big[\big(((x \le a) \wedge (y \le a) \wedge (x \cap y = \mathsf{0})\big) \implies \big(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\big)\Big]$$

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We start with $I_{-1}(a)$, which denotes $a = 0$

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and $\operatorname{conn}(a)$ says that a is connected:

$$(\forall x)(\forall y)\bigl[\bigl((x \cap y = \mathsf{o}) \wedge (x \cup y = a)\bigr) \implies \bigl((x = \mathsf{o}) \vee (x = a)\bigr)\bigr],$$

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- $\operatorname{Dg} X \le n$ iff $2^X \vDash \Delta_n(X)$

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Proof.

Both directions use swelling and shrinking to replace the finite families by combinatorially equivalent subfamilies of the base. $\square$

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Induction: given A and B expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between A' and B', such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As $\mathcal{B}_L$ is a base for the closed sets of L we know, by inductive assumption, that $\operatorname{Ind} L \leq n - 1$. □

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No equivalence, see later.

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Proof.

Let $X = [0,1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0,q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p,1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).

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$\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously but $\operatorname{Dg}[0,1] = 1$. □

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By elementarity we see that $2^X \vDash \delta_n$ iff $L \vDash \delta_n$. Previous theorem: L satisfies δ_n iff 2^{wL} does.
It follows that $\dim X \leq n$ iff $\dim wL \leq n$ for all n. □

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By elementarity we see that $2^X \vDash I_n(X)$ iff $L \vDash I_n(X)$. By previous theorem we know $\operatorname{Ind} wL \le n$, whenever L satisfies $I_n(wL)$.
Thus: $\operatorname{Ind} X \le n$ implies $\operatorname{Ind} wL \le n$. $\square$

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By elementarity we see that $2^X \vDash \Delta_n(X)$ iff $L \vDash \Delta_n(X)$.
By previous theorem we know nothing yet about $\operatorname{Dg} wL$.

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Let A and B be closed and disjoint in wL. Wlog: $A, B \in L$.
Elementarity: there is $C \in L$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leq n - 1$.

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Inductive assumption: $\operatorname{Dg} C \leq n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.

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Inductive assumption: $\operatorname{Dg} C \leq n-1$ in wL, *because* $M = \{D \in L : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-wL is wM.
Still to show: C-in-wL is a cut between A and B in wL. □

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Find H in L around F, disjoint from C.
Back in X no component of H meets C, hence it does *not* meet both A and B.
By well-known topology and elementarity there are disjoint elements H_A and H_B of L such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.
Down in wL we have exactly the same relations, so H_A and H_B show F is not connected.

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There are X with $\dim X < \operatorname{Dg} X$, so $\operatorname{Dg} wL < \operatorname{Dg} X$ and $\operatorname{Ind} wL < \operatorname{Ind} X$ are possible.

Outline

The result

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- X and Y are not homeomorphic

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\bigl((x \sqcap y = 0) \rightarrow ((x \leq u) \wedge (y \leq v) \wedge (u \sqcap v = 0) \wedge (u \sqcup v = 1))\bigr)$$

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In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).
By *compactness*, if some base satisfies this sentence then the space is zero-dimensional.

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If some base satisfies this sentence then the space has no isolated points.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.

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Thus: if X looks like C then X is homeomorphic to C.

The Cantor set is categorical among compact metric spaces.

What the main result says

Among metric continua there is no categorical space.

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Among metric continua there is no categorical space.
No (in)finite list of first-order properties will characterize a single metric continuum.

A case in point: the pseudoarc

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Chainability is *not* first-order.
(Hereditary indecomposability is.)

An embedding lemma

Lemma

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
There is an embedding of $\mathcal{C}$ into the ultrapower of $\mathcal{B}$ by u.

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Let Y be the Wallman space of $\mathcal{D}$.

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- $\mathcal{D}$ is elementarily equivalent to $\mathcal{B}_u$ and hence to $\mathcal{B}$.
- Y maps onto Z (because $\varphi[\mathcal{C}]$ is embedded into $\mathcal{D}$).

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Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and such that Y maps onto Z.
So: Y is not homeomorphic to X.

Outline

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Definition

A continuum, X, is chainable if every (finite) open cover $\mathcal{U}$ has an open chain-refinement $\mathcal{V}$.

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$[0, 1]$ is chainable; the circle S^1 is not.

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A continuum, X, has xxx span zero if every subcontinuum Z of $X \times X$ that satisfies yyy intersects the diagonal $\{\langle x, x\rangle : x \in X\}$.

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$[0, 1]$ has all spans zero, S^1 has all spans non-zero

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In a chainable continuum all spans are zero.

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Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory.
We free it from the metric constraints.

Reflection

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Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

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Proof.

Let X be a counterexample, let $L \prec 2^X$. Then wL is a metrizable counterexample. □

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Not quite ...

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It suffices to consider four-element open covers only.

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Proof for Chainability.

Chainability is now first-order and, like covering dimension, one needs only consider covers and refinements that belong to a certain base. □

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For the converse ...

Span zero, continued

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... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK}.

Span zero, the real argument

Apply Shelah's Ultrapower theorem

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Apply Shelah's Ultrapower theorem: take a cardinal κ, an ultrafilter u on κ and an isomorphism $h : \prod_u (2^{X\times X}) \to \prod_u wK$ (which can be taken to be the identity on K).

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For that we need some topology.

Dualizing ultrapowers

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The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_\kappa^{\leftarrow}(u)$. Bankston calls this the ultracopower of Y; we write Y_u.

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Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

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First expand the language of lattice with two function symbols π_1 and π_2.

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Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.

Outline

Light reading

Website: `fa.its.tudelft.nl/~hart`

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Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.

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There is no categorical metric continuum, to appear.

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