Algebraic Topology, of sorts. Part I

Non impeditus ab ulla scientia

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Outline

Space to lattice

Take a topological space X; it comes with a lattice: 2^X, the family of closed sets, with $\cap$ and $\cup$ as its operations.

This lattice is *distributive* with 0 and 1.

Duality for compact Hausdorff spaces and lattices
What's the use?
Reflections on dimension
Sources
Wallman's construction
Duality

Lattice to space

Let L be a distributive lattice with 0 and 1.

Can we find a space to go with L?

Yes we can! (To quote Bob the Builder)

K. P. Hart Algebraic Topology, of sorts. Part I

Ultrafilters

A *filter* on L is a nonempty subset u that satisfies

- $0 \notin u$
- if $x, y \in u$ then $x \wedge y \in u$
- if $x \in u$ and $y \geqslant x$ then $y \in u$

An *ultrafilter* on L is a filter that is maximal in the congeries of all filters, ordered by inclusion.

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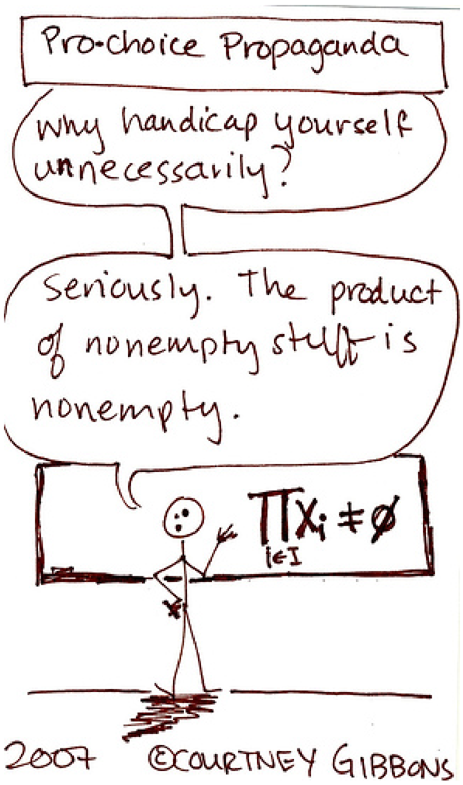
As to their existence . . .

I am definitely

PRO-CHOICE



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Wallman space

The *Wallman space* of L, denoted wL, is defined as follows

- the points are the ultrafilters on L
- the sets $a^* = \{u \in wL : a \in u\}$, where $a \in L$, serve as a base for the *closed* sets

Properties

- $(a \wedge b)^* = a^* \cap b^*$ and $(a \vee b)^* = a^* \cup b^*$ (the latter needs 'ultra', or rather 'prime')
- wL is compact
- points are closed (wL is a T_1-space): $\{u\} = \bigcap\{a^* : a \in u\}$
- $a \mapsto a^*$ is not always injective, e.g., if $L = [\sqrt{2}, \pi]$ then wL consists of just one point
- isomorphism iff $a \not\leqslant b$ implies there is $c > 0$ such that $c \leqslant a$ and $c \wedge b = 0$ (L is said to be separative)

Hausdorff

The space wL is *Hausdorff* iff L is *normal*, i.e.,

$a \wedge b = 0$ implies there are p and q such that

$$a \wedge p = 0, \quad b \wedge q = 0, \text{ and } p \vee q = 1$$

Think of normality for topological spaces formulated in terms of closed sets only
(the complements of p and q are disjoint neighbourhoods of a and b respectively).

From now on: all spaces compact Hausdorff and
all lattices distributive, separative, normal and with 0 and 1.

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Duality?

Clearly $X = w(2^X)$:

X and $w(2^X)$ both correspond to the set of atoms of 2^X.

Also: a continuous $f : X \to Y$ determines a homomorphism $2^f : 2^Y \to 2^X$ by taking preimages: $2^f(a) = f^{-1}[a]$.

If f is injective then 2^f is surjective

If f is surjective then 2^f is injective

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Duality?

Clearly, $L = 2^{wL}$, not!

L and $\{a^* : a \in L\}$ *are* isomorphic, but ...

... $\{a^* : a \in L\}$ is hardly ever the full family of closed sets of wL.

Example

Let L be the lattice generated by the intervals in $[0,1]$ with rational end points. Then $wL = [0,1]$, but L is countable and $2^{[0,1]}$ is not.

Duality?

In fact

Theorem

If $\mathcal{B}$ is a sublattice of 2^X that is also a base for the closed sets of X then $X = w\mathcal{B}$.

So, every X generally corresponds to many lattices.

E.g., if X is compact metric then X corresponds to 2^X but also to (various) countable lattices.

One space, three lattices (at least)

Consider the unit interval $[0,1]$. Here are three lattices for it:

- $2^{[0,1]}$, full lattice of closed sets
- L, generated by intervals with rational end points
- M, generated by $\{[0,q] : q \in \mathbb{Q} \cap (0,1]\}$ together with $\{[\pi q, 1] : q \in \mathbb{Q} \cap [0, 1/\pi)\}$.

Spot the differences

Cardinalities, (non-)existence of atoms ...

Duality?

If $\varphi : L \to M$ is a homomorphism then it induces a continuous map $w\varphi : wM \to wL$:

Let $u \in wM$, then $\{w\varphi(u)\} = \bigcap\{a^* : a \in L, \varphi(a) \in u\}$.

If φ is surjective then $w\varphi$ is injective

If φ is injective then $w\varphi$ is surjective

Homeomorphisms

To begin

Theorem

If two spaces have isomorphic lattice-bases for the closed sets then they are homeomorphic.

Why is this useful?

Spaces have too many points . . .

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The Cantor set

Theorem (Brouwer)

The Cantor set is the only compact metric space that is zero-dimensional and perfect.

- zero-dimensional: the clopen sets form a base
- perfect: no isolated points

Too many points: continuum is way too many.

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The Cantor set

The conditions imply that in any such space the clopen sets form a *countable* atomless Boolean algebra that is also a base for the closed sets.

Any two such Boolean algebras are isomorphic, hence the corresponding spaces are homeomorphic.

The isomorphism can be constructed in a comfortably short recursion along the natural numbers.

Duality for compact Hausdorff spaces and lattices
What's the use?
Reflections on dimension
Sources
Homeomorphisms
Embeddings
Onto mappings

Urysohn's embedding theorem

Theorem

Every compact metric space can be embedded into the Hilbert cube $[0,1]^{\mathbb{N}}$*.*

Let X be compact metrizable, with a countable base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for its closed sets. Take a metric d on X bounded by 1.
As a base for the closed sets of $[0,1]^{\mathbb{N}}$ we take the lattice $\mathcal{L}$ generated by the strips $S_{n,q} = \pi_n^{-1}\big[[0,q]\big]$ and $T_{n,q} = \pi_n^{-1}\big[[q,1]\big]$, where $n \in \mathbb{N}$ and $q \in \mathbb{Q}$.

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Urysohn's embedding theorem

The strips are independent enough to ensure the existence of a homomorphism

$$\varphi : \mathcal{L} \to 2^X$$

that satisfies $\varphi : S_{n,q} \mapsto \{x : d(x, B_n) \leqslant q\}$ and $\varphi : T_{n,q} \mapsto \{x : d(x, B_n) \geqslant q\}$.

Thus, $\varphi[\mathcal{L}]$ is a lattice-base for the closed sets of X.

Apply duality: $w\varphi : X \to [0,1]^{\mathbb{N}}$ is an embedding.

The Cantor set

A recursion, similar to that in the case of homeomorphisms, will produce an injective homomorphism from a given countable lattice into the clopen algebra of the Cantor set.

And so

Theorem (Alexandroff/Hausdorff)

Every compact metric space is a continuous image of the Cantor set.

Covering dimension

Definition (Lebesgue)

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

There is a convenient characterization.

Theorem (Hemmingsen)

$\dim X \leqslant n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

Duality for compact Hausdorff spaces and lattices
What's the use?
Reflections on dimension
Sources
Dimension functions
Formulas
Bases
Reflections

Covering dimension

We say $\dim X = n$ if $\dim X \leqslant n$ but $\dim X \nleqslant n-1$;
also, $\dim X = \infty$ means $\dim X \nleqslant n$ for all $n \in \mathbb{N}$.
$\dim X$ is the *covering dimension* of X.

Theorem

$\dim[0,1]^n = n$ for all $n \in \mathbb{N} \cup \{\infty\}$.

Thus, dim helps in showing that all cubes are topologically distinct.

Large inductive dimension

Definition (Čech)

$\operatorname{Ind} X \leqslant n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\operatorname{Ind} L \leqslant n-1$.
The starting point: $\operatorname{Ind} X \leqslant -1$ iff $X = \emptyset$.

L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

Large inductive dimension

We say $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leqslant n$ but $\operatorname{Ind} X \nleqslant n-1$;
also, $\operatorname{Ind} X = \infty$ means $\operatorname{Ind} X \nleqslant n$ for all $n \in \mathbb{N}$.
$\operatorname{Ind} X$ is the *large inductive dimension* of X.

Theorem

$\operatorname{Ind}[0,1]^n = n$ *for all* $n \in \mathbb{N} \cup \{\infty\}$.

Thus, Ind helps in showing that all cubes are topologically distinct.

Dimensionsgrad

Definition (Brouwer)

$\operatorname{Dg} X \leqslant n$ if between every two disjoint closed sets A and B there is a cut C that satisfies $\operatorname{Dg} C \leqslant n-1$.
The starting point: $\operatorname{Dg} X \leqslant -1$ iff $X = \emptyset$.

C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

Duality for compact Hausdorff spaces and lattices
What's the use?
Reflections on dimension
Sources
Dimension functions
Formulas
Bases
Reflections

Dimensionsgrad

We say $\operatorname{Dg} X = n$ if $\operatorname{Dg} X \leqslant n$ but $\operatorname{Dg} X \not\leqslant n-1$;
also, $\operatorname{Dg} X = \infty$ means $\operatorname{Dg} X \not\leqslant n$ for all $n \in \mathbb{N}$.
$\operatorname{Dg} X$ is the *Dimensionsgrad* of X.

Theorem

$\operatorname{Dg}[0,1]^n = n$ for all $n \in \mathbb{N} \cup \{\infty\}$.

Thus, Dg helps in showing that all cubes are topologically distinct.

Equalities

Theorem

For every compact metrizable space X we have

$$\dim X = \operatorname{Dg} X = \operatorname{Ind} X$$

- $\dim X = \operatorname{Ind} X$ for all metrizable X
- $\dim X = \operatorname{Dg} X$ for all σ-compact metrizable X ...
- ... but not for all separable metrizable X

More inequalities

For compact Hausdorff spaces:

- $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ (each partition is a cut)
- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)
- $\dim X \leqslant \operatorname{Dg} X$ (Fedorchuk)

We will (re)prove the last two inequalities algebraically.

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets and cast as a formula, δ_n, in the language of lattices

$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2})\\
&\qquad\big[(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = 0) \to\\
&\qquad\big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \cdots \wedge (x_{n+2} \leqslant y_{n+2})\\
&\qquad\wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = 0)\\
&\qquad\wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1)\big)\big].
\end{aligned}
$$

Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$
\begin{aligned}
&(\forall x)(\forall y)(\exists u)\\
&\big[\big(((x \leqslant a)\wedge(y \leqslant a)\wedge(x\cap y = 0)\big) \to \big(\operatorname{partn}(u,x,y,a)\wedge I_{n-1}(u)\big)\big]
\end{aligned}
$$

where $\operatorname{partn}(u,x,y,a)$ says that u is a partition between x and y in the (sub)space a:

$$(\exists f)(\exists g)\big((x \cap f = 0) \wedge (y \cap g = 0) \wedge (f \cup g = a) \wedge (f \cap g = u)\big).$$

We start with $I_{-1}(a)$, which denotes $a = 0$

Dimensionsgrad

Here we have the recursive definition of a formula $\Delta_n(a)$:

$$(\forall x)(\forall y)(\exists u)$$
$$\big[\big((x \leqslant a) \wedge (y \leqslant a) \wedge (x \cap y = 0)\big) \to \big(\mathrm{cut}(u, x, y, a) \wedge \Delta_{n-1}(u)\big)\big],$$

and $\Delta_{-1}(a)$ denotes $a = 0$.

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Sources
Dimension functions
Formulas
Bases
Reflections

Dimensionsgrad (auxiliary formulas)

The formula $\mathrm{cut}(u, x, y, a)$ expresses that u is a cut between x and y in a:

$$(\forall v)\big[\big((v \leqslant a) \wedge \mathrm{conn}(v) \wedge (v \cap x \neq 0) \wedge (v \cap y \neq 0)\big) \to (v \cap u \neq 0)\big],$$

and $\mathrm{conn}(a)$ says that a is connected:

$$(\forall x)(\forall y)\big[\big((x \cap y = 0) \wedge (x \cup y = a)\big) \to \big((x = 0) \vee (x = a)\big)\big],$$

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Algebraic Topology, of sorts. Part I

Wherefore formulas?

Romeo and Juliet, Act 2, scene 2 (alternate):
O Formulas, Formulas! — Wherefore useth thou Formulas?

Forsooth! It giveth us an algebraic handle on these dimensions:

- $\dim X \leqslant n$ iff δ_n holds in 2^X
- $\operatorname{Ind} X \leqslant n$ iff $I_n(X)$ holds in 2^X
- $\operatorname{Dg} X \leqslant n$ iff $\Delta_n(X)$ holds in 2^X

Covering dimension

Theorem

Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Proof: compactness and a shrinking-and-swelling argument.

Large inductive dimension

Theorem

Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

Proof: induction and, again, a swelling-and-shrinking argument.

No equivalence, see later.

Dimensionsgrad

Theorem

Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Proof: we can cheat and create, for $[0, 1]$ say, a lattice base without connected elements; that base satisfies $\Delta_0(X)$ vacuously.

Take a rich sublattice

Let X be compact Hausdorff and let $\mathcal{B}$ be a countable sublattice of 2^X with exactly the same algebraic properties as 2^X.

If you know your model theory: apply the Löwenheim-Skolem theorem.
If not: think of taking a countable algebraic subfield of $\mathbb{C}$, say.

Duality for compact Hausdorff spaces and lattices
What's the use?
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Sources
Dimension functions
Formulas
Bases
Reflections

Covering dimension vs large inductive dimension

The formula δ_n holds in $\mathcal{B}$ iff it holds in 2^X, hence

$$\dim w\mathcal{B} = \dim X.$$

The formula $I_n(X)$ holds in $\mathcal{B}$ iff it holds in 2^X, hence

$$\operatorname{Ind} w\mathcal{B} \leqslant \operatorname{Ind} X.$$

But $w\mathcal{B}$ is compact metrizable, so $\dim w\mathcal{B} = \operatorname{Ind} w\mathcal{B}$, hence

$$\dim X \leqslant \operatorname{Ind} X.$$

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Covering dimension vs large inductive dimension

There are (many) compact Hausdorff spaces with non-coinciding dimensions, e.g., an early example of a compact L such that $\dim L = 1$ and $\operatorname{Ind} L = 2$ (Lokucievskiĭ).

In that case $\operatorname{Ind} w\mathcal{B} < \operatorname{Ind} L$ for countable (rich) sublattices of 2^L.

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Covering dimension vs Dimensionsgrad

The stronger inequality $\dim X \leqslant \operatorname{Dg} X$ can be proved via $w\mathcal{B}$ as well.
The argument is more involved.
It uses in an essential way that $\mathcal{B}$ is a rich sublattice of 2^X.

I'll spare you the details.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart.
Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.