Algebraic Topology, of sorts. Part I

Non impeditus ab ulla scientia

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Galway in Brum, 13 July, 2010: 14:00 – 15:00

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Outline

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Space to lattice

Take a topological space X

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Take a topological space X; it comes with a lattice: 2^X, the family of closed sets

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Take a topological space X; it comes with a lattice: 2^X, the family of closed sets, with $\cap$ and $\cup$ as its operations.

This lattice is *distributive* with 0 and 1.

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Can we find a space to go with L?

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Yes we can!

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Yes we can! (To quote Bob the Builder)

Outline

Ultrafilters

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An *ultrafilter* on L is a filter that is maximal in the congeries of all filters, ordered by inclusion.

As to their existence . . .

I am definitely

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PRO-CHOICE

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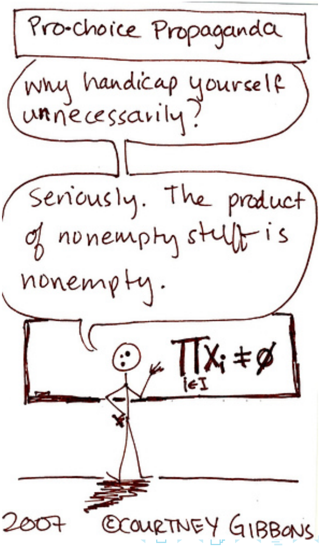

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- the points are the ultrafilters on L
- the sets $a^* = \{u \in wL : a \in u\}$, where $a \in L$, serve as a base for the *closed* sets

Properties

- $(a \wedge b)^* = a^* \cap b^*$ and $(a \vee b)^* = a^* \cup b^*$ (the latter needs 'ultra', or rather 'prime')

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  if $L = [\sqrt{2}, \pi]$ then wL consists of just one point
- isomorphism iff
  $a \nleqslant b$ implies there is $c > 0$ such that $c \leqslant a$ and $c \wedge b = 0$
  (L is said to be separative)

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From now on: all spaces compact Hausdorff and
all lattices distributive, separative, normal and with 0 and 1.

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Let L be the lattice generated by the intervals in $[0, 1]$ with rational end points. Then $wL = [0, 1]$, but L is countable and $2^{[0,1]}$ is not.

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E.g., if X is compact metric then X corresponds to 2^X but also to (various) countable lattices.

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Cardinalities, (non-)existence of atoms ...

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Spaces have too many points . . .

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The Cantor set

Theorem (Brouwer)

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Too many points: continuum is way too many.

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The isomorphism can be constructed in a comfortably short recursion along the natural numbers.

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Urysohn's embedding theorem

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Let X be compact metrizable, with a countable base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for its closed sets. Take a metric d on X bounded by 1.

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Let X be compact metrizable, with a countable base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for its closed sets. Take a metric d on X bounded by 1.

As a base for the closed sets of $[0,1]^{\mathbb{N}}$ we take the lattice $\mathcal{L}$ generated by the strips $S_{n,q} = \pi_n^{-1}\big[[0,q]\big]$ and $T_{n,q} = \pi_n^{-1}\big[[q,1]\big]$, where $n \in \mathbb{N}$ and $q \in \mathbb{Q}$.

Urysohn's embedding theorem

The strips are independent enough to ensure the existence of a homomorphism

$$\varphi : \mathcal{L} \to 2^X$$

that satisfies $\varphi : S_{n,q} \mapsto \{x : d(x, B_n) \leqslant q\}$ and $\varphi : T_{n,q} \mapsto \{x : d(x, B_n) \geqslant q\}$.

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Apply duality: $w\varphi : X \to [0, 1]^{\mathbb{N}}$ is an embedding.

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Theorem (Alexandroff/Hausdorff)

Every compact metric space is a continuous image of the Cantor set.

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Covering dimension

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Theorem (Hemmingsen)

$\dim X \leqslant n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

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Thus, dim helps in showing that all cubes are topologically distinct.

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L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

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C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

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- $\dim X = \operatorname{Ind} X$ for all metrizable X
- $\dim X = \operatorname{Dg} X$ for all σ-compact metrizable X ...

Equalities

Theorem

For every compact metrizable space X we have

$$\dim X = \operatorname{Dg} X = \operatorname{Ind} X$$

- $\dim X = \operatorname{Ind} X$ for all metrizable X
- $\dim X = \operatorname{Dg} X$ for all σ-compact metrizable X ...
- ... but not for all separable metrizable X

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More inequalities

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- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)
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We will (re)prove the last two inequalities algebraically.

Outline

Covering dimension

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$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2})\\
&\qquad\Big[(x_1 \cap x_2 \cap \cdots \cap x_{n+2} = 0) \to\\
&\qquad\Big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \cdots \wedge (x_{n+2} \leqslant y_{n+2})\\
&\qquad\wedge (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = 0)\\
&\qquad\wedge (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1)\Big)\Big].
\end{aligned}
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where $\operatorname{partn}(u, x, y, a)$ says that u is a partition between x and y in the (sub)space a:

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We start with $I_{-1}(a)$, which denotes $a = \mathfrak{o}$

Dimensionsgrad

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and $\mathrm{conn}(a)$ says that a is connected:

$$(\forall x)(\forall y)\big[((x \cap y = 0) \wedge (x \cup y = a)) \to ((x = 0) \vee (x = a))\big],$$

Wherefore formulas?

Romeo and Juliet, Act 2, scene 2 (alternate)

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- $\operatorname{Dg} X \leqslant n$ iff $\Delta_n(X)$ holds in 2^X

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Outline

Covering dimension

Theorem

Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Proof: compactness and a shrinking-and-swelling argument.

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Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

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Theorem

Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

Proof: induction and, again, a swelling-and-shrinking argument.

No equivalence, see later.

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Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

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Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Proof: we can cheat and create, for $[0,1]$ say, a lattice base without connected elements; that base satisfies $\Delta_0(X)$ vacuously.

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Outline

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Take a rich sublattice

Let X be compact Hausdorff and let $\mathcal{B}$ be a countable sublattice of 2^X with exactly the same algebraic properties as 2^X.

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If you know your model theory: apply the Löwenheim-Skolem theorem.
If not: think of taking a countable algebraic subfield of $\mathbb{C}$, say.

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But $w\mathcal{B}$ is compact metrizable, so $\dim w\mathcal{B} = \operatorname{Ind} w\mathcal{B}$, hence

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$$\dim X \leqslant \operatorname{Ind} X.$$

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Covering dimension vs large inductive dimension

There are (many) compact Hausdorff spaces with non-coinciding dimensions, e.g., an early example of a compact L such that $\dim L = 1$ and $\operatorname{Ind} L = 2$ (Lokucievskiĭ).

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There are (many) compact Hausdorff spaces with non-coinciding dimensions, e.g., an early example of a compact L such that $\dim L = 1$ and $\operatorname{Ind} L = 2$ (Lokucievskiĭ).

In that case $\operatorname{Ind} w\mathcal{B} < \operatorname{Ind} L$ for countable (rich) sublattices of 2^L.

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The stronger inequality $\dim X \leqslant \mathrm{Dg}\, X$ can be proved via $w\mathcal{B}$ as well.

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The stronger inequality $\dim X \leqslant \operatorname{Dg} X$ can be proved via $w\mathcal{B}$ as well.
The argument is more involved.
It uses in an essential way that $\mathcal{B}$ is a rich sublattice of 2^X.

I'll spare you the details.

Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart.
Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.