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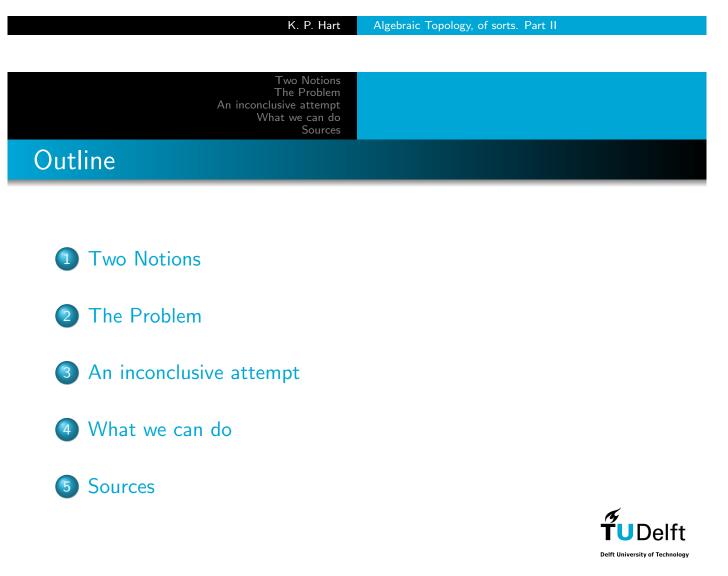
Algebraic Topology, of sorts. Part II Non impeditus ab ulla scientia

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Galway in Brum, 15 July, 2010: 10:00 - 11:00





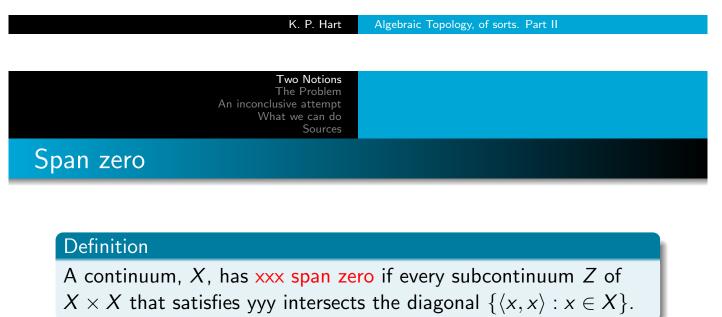
Chainability

Definition

A continuum, X, is chainable if every (finite) open cover \mathcal{U} has an open chain-refinement \mathcal{V} , i.e., \mathcal{V} can be written as $\{V_i : i < n\}$ such that $V_i \cap V_j \neq \emptyset$ iff $|i - j| \leq 1$.

[0,1] is chainable; the circle S^1 is not.





XXX	ууу	symbol
	$\pi_1[Z] = \pi_2[Z]$	σX
semi	$\pi_1[Z]\subseteq\pi_2[Z]$	$\frac{1}{2}\sigma X$
surjective	$\pi_1[Z] = \pi_2[Z] = X$	$s\sigma X$
surjective semi	$\pi_2[Z] = X$	$s\frac{1}{2}\sigma X$
$[0,1]$ has all spans zero, S^1	^L has all spans non-zero	<u>(4</u>



The problem

Theorem

In a chainable continuum all spans are zero.

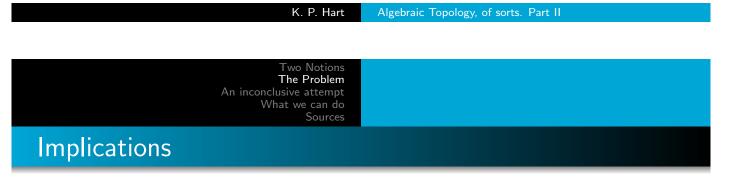
Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory.

But it makes non-metric sense as well.





$$\sigma X = 0 \quad \leftarrow \quad \frac{1}{2}\sigma X = 0$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$s\sigma X = 0 \quad \leftarrow \quad s\frac{1}{2}\sigma X = 0$$

or, contrapositively



 \mathbb{H}^* is not chainable

 $\mathbb{H} = [0, \infty)$ and \mathbb{H}^* is its Čech-Stone remainder. For i = 0, 1, 2, 3 put

$$U_i = \bigcup_{n=0}^{\infty} \left(4n + i - \frac{5}{8}, 4n + i + \frac{5}{8} \right)$$

 and

$$O_i = \mathsf{Ex} \, U_i \cap \mathbb{H}^*$$

where $\mathsf{Ex} U = \beta \mathbb{H} \setminus \mathsf{cl}(\mathbb{H} \setminus U)$ (the largest open set in $\beta \mathbb{H}$ that intersects \mathbb{H} in U).



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The open cover $\{O_0, O_1, O_2, O_3\}$ of \mathbb{H}^* does not have a chain refinement — nice exercise, but a bit convoluted.



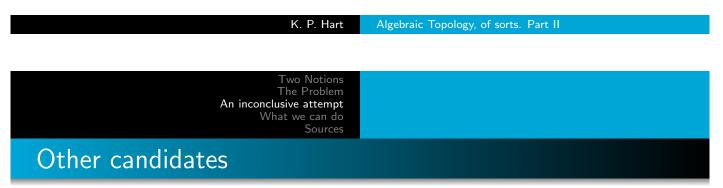


It would be nice if some of the spans of \mathbb{H}^* were zero: we'd have a non-metric counterexample to Lelek's conjecture.

However: consider $f : \mathbb{H} \to \mathbb{H}$, defined by f(x) = x + 1, and its extension $\beta f : \beta \mathbb{H} \to \beta \mathbb{H}$, and that extension's restriction $f^* : \mathbb{H}^* \to \mathbb{H}^*$.

Its graph witnesses that the surjective span of \mathbb{H}^* is non-zero and hence so are the other three.





Consider $\mathbb{M} = \omega \times [0, 1]$ and its Čech-Stone compactification $\beta \mathbb{M}$.

The extension $\beta \pi : \beta \mathbb{M} \to \beta \omega$ of the projection $\pi : \mathbb{M} \to \omega$ divides $\beta \mathbb{M}$ into continua.

For $u \in \omega^*$ we punt $\mathbb{I}_u = \beta \pi^{\leftarrow}(u)$.

What can we say about the spans of the \mathbb{I}_u ?



The span of \mathbb{I}_u

Theorem

The span of \mathbb{I}_u is non-zero.

The proof is like that for \mathbb{H}^* : the continua \mathbb{I}_u contain subcontinua that are quite similar to \mathbb{H}^* and they allow an analogue of the graph of $x \mapsto x + 1$.



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The other spans of \mathbb{I}_u	

Theorem (CH)

The surjective span of \mathbb{I}_u is non-zero.

The proof is more involved and can best be illustrated with a picture.



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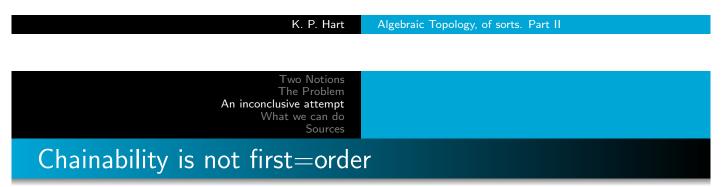
 \mathbb{I}_u has a (very) nice base for its closed sets: the *ultrapower* of $2^{\mathbb{I}}$ by the ultrafilter u.

Ultrapower of a lattice L by an ultrafilter u on a cardinal κ :

 L^{κ}/\sim_{u}

where $f \sim_u g$ means $\{\alpha : f(\alpha) = g(\alpha)\} \in u$. L and L^{κ}/\sim_u have exactly the same algebraic properties.





Chainability is, just like covering dimension, a property of every/some lattice base for the closed sets. (Shrink-and-swell again.)

Now then, ..., $2^{\mathbb{I}}$ satisfies 'chainability' but L^{κ}/\sim_{u} does not, so

unlike the dimensions, chainability is not expressible in first-order terms.



The natural formulation is an $L_{\omega_1,\omega}$ -formula.

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$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)$$
$$((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \rightarrow \bigvee_{n \in \omega} \Phi_n(u_1, u_2, u_3, u_4))$$

where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an *n*-element chain refinement.

It (indeed) suffices to consider four-element open covers only.



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Span zero is	

The status of span zero is not clear: it is either

- not a property reducible to bases or
- not first-order.



Reflection

Theorem

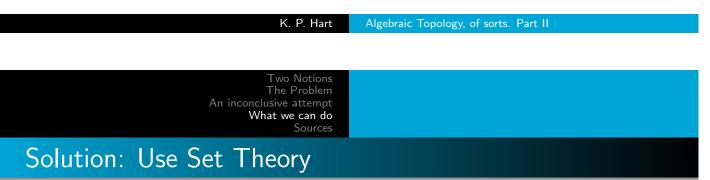
Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

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Р	roof.	

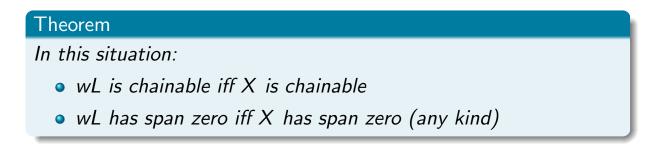
Let X be a counterexample, let $L \prec 2^X$ (an elementary sublattice). Then wL is a metrizable counterexample.

Not quite ... because of what we have just seen.





Let θ be 'suitably large' and let $M \prec H(\theta)$ be a countable elementary substructure and let $L = M \cap 2^{X}$.





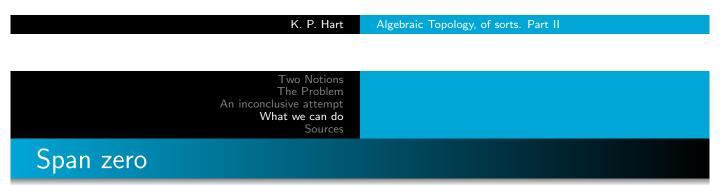
Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.

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Furthermore, one needs only consider covers and refinements that belong to a certain base.





Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$.

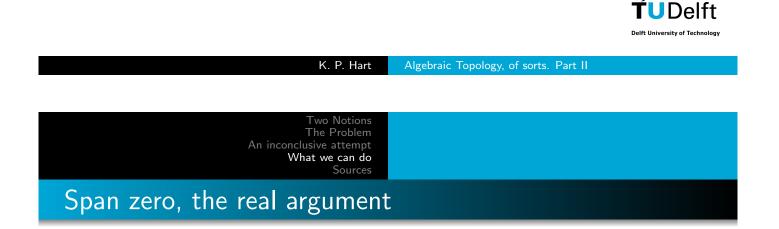
This gives the easy part: if there is a 'bad' continuum in $X \times X$ then there is one in M and it is equally bad in $wL \times wL$.

For the converse ...



... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK} .



Apply Shelah's Ultrapower theorem: take a cardinal κ , an ultrafilter u on κ and an isomorphism $h: \prod_{u} (2^{X \times X}) \to \prod_{u} wK$ (which can be taken to be the identity on K).

How does that help?

For that we need some topology.



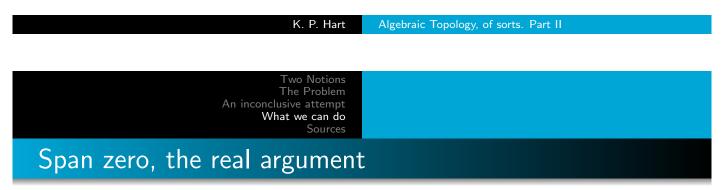
Take a compact Hausdorff space Y with a lattice base B. Also take a cardinal κ and an ultrafilter u on κ .

Consider $\beta(\kappa \times Y)$. We have two maps

- $p_{\kappa} : \beta(\kappa \times Y) \to \beta \kappa$ (the extension of $\langle \alpha, y \rangle \mapsto \alpha$).
- $p_Y : \beta(\kappa \times Y) \to \beta \kappa$ (the extension of $\langle \alpha, y \rangle \mapsto y$).

The Wallman space of the ultrapower $\prod_{u} B$ is the fiber $p_{\kappa}^{\leftarrow}(u)$. Bankston calls this the ultracopower of Y; we write Y_{u} .





Back to $Z \subseteq wK$.

- Let $Z_u = \operatorname{cl}(\kappa \times Z) \cap p_{\kappa}^{\leftarrow}(u)$.
- Z_u is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ (wh is dual to h).
- $Z_X = p_{X \times X} [wh[Z_u]]$ is a continuum in $X \times X$.
- And

$$q_{\mathcal{K}}[Z_X] = q_{\mathcal{K}}\Big[p_{X\times X}\big[wh[Z_u]\big]\Big] = p_{w\mathcal{K}}\Big[(wh)^{-1}\big[wh[Z_u]\big]\Big] = Z$$

So, that's it!? Almost.

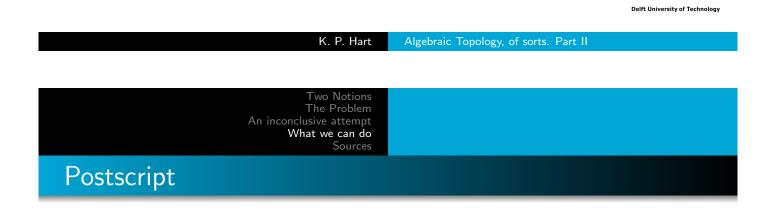


Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2 .

Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.



Logan Hoehn has constructed a metrizable continuum that is non-chainable but that has span zero.



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- K. P. Hart, B. van der Steeg,
 Span, chainability and the continua ℍ* and I_u, Topology and its Applications, 151, 1–3 (2005), 226–237.
- D. Bartošová, K. P. Hart, L. Hoehn, B. van der Steeg, Lelek's problem is not a metric problem, to appear.



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