Algebraic Topology, of sorts. Part II

Non impeditus ab ulla scientia

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Outline

1. Two Notions
2. The Problem
3. An inconclusive attempt
4. What we can do
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Chainability

Definition

A continuum, X, is chainable if every (finite) open cover $\mathcal{U}$ has an open chain-refinement $\mathcal{V}$, i.e., $\mathcal{V}$ can be written as $\{V_i : i < n\}$ such that $V_i \cap V_j \neq \emptyset$ iff $|i - j| \leqslant 1$.

$[0, 1]$ is chainable; the circle S^1 is not.

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Span zero

Definition

A continuum, X, has xxx span zero if every subcontinuum Z of $X \times X$ that satisfies yyy intersects the diagonal $\{\langle x, x\rangle : x \in X\}$.

xxx	yyy	symbol
...	$\pi_1[Z] = \pi_2[Z]$	σX
semi	$\pi_1[Z] \subseteq \pi_2[Z]$	$\frac{1}{2}\sigma X$
surjective	$\pi_1[Z] = \pi_2[Z] = X$	$s\sigma X$
surjective semi	$\pi_2[Z] = X$	$s\frac{1}{2}\sigma X$

$[0, 1]$ has all spans zero, S^1 has all spans non-zero

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The problem

Theorem

In a chainable continuum all spans are zero.

Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory.

But it makes non-metric sense as well.

Implications

$$\begin{array}{ccc} \sigma X = 0 & \leftarrow & \frac{1}{2}\sigma X = 0 \\ \downarrow & & \downarrow \\ s\sigma X = 0 & \leftarrow & s\frac{1}{2}\sigma X = 0 \end{array}$$

or, contrapositively

$$\begin{array}{ccc} \sigma X > 0 & \rightarrow & \frac{1}{2}\sigma X > 0 \\ \uparrow & & \uparrow \\ s\sigma X > 0 & \rightarrow & s\frac{1}{2}\sigma X > 0 \end{array}$$

$\mathbb{H}^*$ is not chainable

$\mathbb{H} = [0, \infty)$ and $\mathbb{H}^*$ is its Čech-Stone remainder.
For $i = 0, 1, 2, 3$ put

$$U_i = \bigcup_{n=0}^{\infty} \left(4n + i - \frac{5}{8}, 4n + i + \frac{5}{8}\right)$$

and

$$O_i = \operatorname{Ex} U_i \cap \mathbb{H}^*$$

where $\operatorname{Ex} U = \beta\mathbb{H} \setminus \operatorname{cl}(\mathbb{H} \setminus U)$
(the largest open set in $\beta\mathbb{H}$ that intersects $\mathbb{H}$ in U).

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$\mathbb{H}^*$ is not chainable

The open cover $\{O_0, O_1, O_2, O_3\}$ of $\mathbb{H}^*$ does not have a chain refinement — nice exercise, but a bit convoluted.

The spans of $\mathbb{H}^*$

It would be nice if some of the spans of $\mathbb{H}^*$ were zero:
we'd have a non-metric counterexample to Lelek's conjecture.

However: consider $f : \mathbb{H} \to \mathbb{H}$, defined by $f(x) = x + 1$,
and its extension $\beta f : \beta\mathbb{H} \to \beta\mathbb{H}$,
and that extension's restriction $f^* : \mathbb{H}^* \to \mathbb{H}^*$.

Its graph witnesses that the surjective span of $\mathbb{H}^*$ is non-zero and hence so are the other three.

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Other candidates

Consider $\mathbb{M} = \omega \times [0,1]$ and its Čech-Stone compactification $\beta\mathbb{M}$.

The extension $\beta\pi : \beta\mathbb{M} \to \beta\omega$ of the projection $\pi : \mathbb{M} \to \omega$ divides $\beta\mathbb{M}$ into continua.

For $u \in \omega^*$ we punt $\mathbb{I}_u = \beta\pi^{\leftarrow}(u)$.

What can we say about the spans of the $\mathbb{I}_u$?

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The span of $\mathbb{I}_u$

Theorem

The span of $\mathbb{I}_u$ is non-zero.

The proof is like that for $\mathbb{H}^*$: the continua $\mathbb{I}_u$ contain subcontinua that are quite similar to $\mathbb{H}^*$ and they allow an analogue of the graph of $x \mapsto x + 1$.

The other spans of $\mathbb{I}_u$

Theorem (CH)

The surjective span of $\mathbb{I}_u$ is non-zero.

The proof is more involved and can best be illustrated with a picture.

Why is this interesting?

$\mathbb{I}_u$ has a (very) nice base for its closed sets: the *ultrapower* of $2^{\mathbb{I}}$ by the ultrafilter u.

Ultrapower of a lattice L by an ultrafilter u on a cardinal κ:

$$L^\kappa/{\sim_u}$$

where $f \sim_u g$ means $\{\alpha : f(\alpha) = g(\alpha)\} \in u$.
L and $L^\kappa/{\sim_u}$ have exactly the same algebraic properties.

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Chainability is not first=order

Chainability is, just like covering dimension, a property of every/some lattice base for the closed sets.
(Shrink-and-swell again.)

Now then, ..., $2^{\mathbb{I}}$ satisfies 'chainability' but $L^\kappa/{\sim_u}$ does not, so

unlike the dimensions, chainability is not expressible in first-order terms.

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A formula for chainability

The natural formulation is an $L_{\omega_1,\omega}$-formula.

$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4) \Big((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \to \bigvee_{n\in\omega} \Phi_n(u_1,u_2,u_3,u_4)\Big)$$

where $\Phi_n(u_1,u_2,u_3,u_4)$ expresses that $\{u_1,u_2,u_3,u_4\}$ has an n-element chain refinement.

It (indeed) suffices to consider four-element open covers only.

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Span zero is . . .

The status of span zero is not clear: it is either

- not a property reducible to bases or
- not first-order.

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Reflection

Theorem

Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

Proof.

Let X be a counterexample, let $L \prec 2^X$ (an elementary sublattice). Then wL is a metrizable counterexample. □

Not quite . . . because of what we have just seen.

Solution: Use Set Theory

Let θ be 'suitably large' and let $M \prec H(\theta)$ be a countable elementary substructure and let $L = M \cap 2^X$.

Theorem

In this situation:

- *wL is chainable iff X is chainable*
- *wL has span zero iff X has span zero (any kind)*

Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.

Furthermore, one needs only consider covers and refinements that belong to a certain base.

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Span zero

Key observation: let $K = M \cap 2^{X\times X}$, then $wK = wL \times wL$.

This gives the easy part: if there is a 'bad' continuum in $X \times X$ then there is one in M and it is equally bad in $wL \times wL$.

For the converse . . .

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Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK}.

Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal κ, an ultrafilter u on κ and an isomorphism $h : \prod_u (2^{X \times X}) \to \prod_u wK$ (which can be taken to be the identity on K).

How does that help?

For that we need some topology.

Dualizing ultrapowers

Take a compact Hausdorff space Y with a lattice base B. Also take a cardinal κ and an ultrafilter u on κ.

Consider $\beta(\kappa \times Y)$. We have two maps

- $p_\kappa : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto \alpha$).
- $p_Y : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto y$).

The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_\kappa^\leftarrow(u)$. Bankston calls this the ultracopower of Y; we write Y_u.

Span zero, the real argument

Back to $Z \subseteq wK$.

- Let $Z_u = \mathrm{cl}(\kappa \times Z) \cap p_\kappa^\leftarrow(u)$.
- Z_u is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ (wh is dual to h).
- $Z_X = p_{X\times X}\bigl[wh[Z_u]\bigr]$ is a continuum in $X \times X$.
- And

$$q_K[Z_X] = q_K\Bigl[p_{X\times X}\bigl[wh[Z_u]\bigr]\Bigr] = p_{wK}\Bigl[(wh)^{-1}\bigl[wh[Z_u]\bigr]\Bigr] = Z$$

So, that's it!? Almost.

Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2.

Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.

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Postscript

Logan Hoehn has constructed a metrizable continuum that is non-chainable but that has span zero.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart, B. van der Steeg,
Span, chainability and the continua $\mathbb{H}^*$ and $\mathbb{I}_u$, Topology and its Applications, 151, 1–3 (2005), 226–237.

D. Bartošová, K. P. Hart, L. Hoehn, B. van der Steeg,
Lelek's problem is not a metric problem, to appear.