## Algebraic Topology, of sorts. Part II Non impeditus ab ulla scientia

K. P. Hart

Faculty EEMCS TU Delft

Galway in Brum, 15 July, 2010: 10:00 - 11:00







- 2 The Problem
- 3 An inconclusive attempt
- What we can do





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[0,1] is chainable; the circle  $S^1$  is not.





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A continuum, X, has xxx span zero if every subcontinuum Z of  $X \times X$  that satisfies yyy intersects the diagonal  $\{\langle x, x \rangle : x \in X\}$ .





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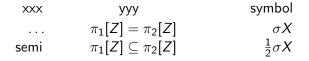
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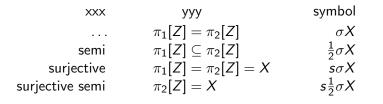
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[0,1] has all spans zero,  $S^1$  has all spans non-zero















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In a chainable continuum all spans are zero.





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What about the converse?

This is an important problem in metric continuum theory.

But it makes non-metric sense as well.





$$\frac{1}{2}\sigma X = 0$$



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## Implications



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where  $\operatorname{Ex} U = \beta \mathbb{H} \setminus \operatorname{cl}(\mathbb{H} \setminus U)$ (the largest open set in  $\beta \mathbb{H}$  that intersects  $\mathbb{H}$  in U).



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# The open cover $\{\mathit{O}_0,\mathit{O}_1,\mathit{O}_2,\mathit{O}_3\}$ of $\mathbb{H}^*$ does not have a chain refinement



#### $\mathbb{H}^*$ is not chainable

# The open cover $\{O_0, O_1, O_2, O_3\}$ of $\mathbb{H}^*$ does not have a chain refinement — nice exercise, but a bit convoluted.



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Its graph witnesses that the surjective span of  $\mathbb{H}^*$  is non-zero and hence so are the other three.

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#### Other candidates

Consider  $\mathbb{M} = \omega \times [0, 1]$  and its Čech-Stone compactification  $\beta \mathbb{M}$ .



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For  $u \in \omega^*$  we punt  $\mathbb{I}_u = \beta \pi^{\leftarrow}(u)$ .

What can we say about the spans of the  $\mathbb{I}_u$ ?



## The span of $\mathbb{I}_u$

#### Theorem

The span of  $\mathbb{I}_u$  is non-zero.



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The proof is like that for  $\mathbb{H}^*$ : the continua  $\mathbb{I}_u$  contain subcontinua that are quite similar to  $\mathbb{H}^*$  and they allow an analogue of the graph of  $x \mapsto x + 1$ .



#### The other spans of $\mathbb{I}_u$

Theorem (CH)

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#### Theorem (CH)

The surjective span of  $\mathbb{I}_u$  is non-zero.

The proof is more involved and can best be illustrated with a picture.



## Why is this interesting?

 $\mathbb{I}_u$  has a (very) nice base for its closed sets:



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Ultrapower of a lattice L by an ultrafilter u on a cardinal  $\kappa$ :

 $L^{\kappa}/\sim_{u}$ 

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Ultrapower of a lattice L by an ultrafilter u on a cardinal  $\kappa$ :

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where  $f \sim_u g$  means  $\{\alpha : f(\alpha) = g(\alpha)\} \in u$ . L and  $L^{\kappa}/\sim_u$  have exactly the same algebraic properties.



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Now then, ...,  $2^{\mathbb{I}}$  satisfies 'chainability' but  $L^{\kappa}/\sim_{u}$  does not, so

unlike the dimensions, chainability is not expressible in first-order terms.



## A formula for chainability

The natural formulation is an  $L_{\omega_1,\omega}$ -formula.



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$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)$$
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where  $\Phi_n(u_1, u_2, u_3, u_4)$  expresses that  $\{u_1, u_2, u_3, u_4\}$  has an *n*-element chain refinement.

It (indeed) suffices to consider four-element open covers only.



#### The status of span zero is not clear: it is either



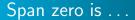
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The status of span zero is not clear: it is either

- not a property reducible to bases or
- not first-order.









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## Reflection

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Not quite ... because of what we have just seen.

#### Solution: Use Set Theory

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In this situation:

• wL is chainable iff X is chainable



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Let  $\theta$  be 'suitably large' and let  $M \prec H(\theta)$  be a countable elementary substructure and let  $L = M \cap 2^X$ .

#### Theorem

In this situation:

- wL is chainable iff X is chainable
- wL has span zero iff X has span zero (any kind)



# Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of  $2^X$  and finite ordinals.



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Chainability is now first-order; we can quantify over the finite subsets of  $2^X$  and finite ordinals.

Furthermore, one needs only consider covers and refinements that belong to a certain base.





#### Key observation: let $K = M \cap 2^{X \times X}$ , then $wK = wL \times wL$ .



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#### Span zero

Key observation: let  $K = M \cap 2^{X \times X}$ , then  $wK = wL \times wL$ .

This gives the easy part: if there is a 'bad' continuum in  $X \times X$  then there is one in M and it is equally bad in  $wL \times wL$ .



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For the converse ...



# Span zero, continued

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# Span zero, continued

... if  $Z \subseteq wL \times wL$  is 'bad' then there is an equally bad continuum in  $X \times X$  that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of  $2^{wK}$ .



#### Span zero, the real argument

Apply Shelah's Ultrapower theorem



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# Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal  $\kappa$ , an ultrafilter u on  $\kappa$  and an isomorphism  $h: \prod_u (2^{X \times X}) \to \prod_u wK$  (which can be taken to be the identity on K).



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For that we need some topology.



# Dualizing ultrapowers

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The Wallman space of the ultrapower  $\prod_{u} B$  is the fiber  $p_{\kappa}^{\leftarrow}(u)$ . Bankston calls this the ultracopower of Y; we write  $Y_{u}$ .



#### Span zero, the real argument

Back to  $Z \subseteq wK$ .



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# Span zero, the real argument

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• Let  $Z_u = cl(\kappa \times Z) \cap p_{\kappa}^{\leftarrow}(u)$ .



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So, that's it!? Almost.



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Apply Shelah's theorem with this extended language. Then  $Z_X$  will inherit the mapping properties that Z has.

Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.

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Logan Hoehn has constructed a metrizable continuum that is non-chainable but that has span zero.



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- 2 The Problem
- 3 An inconclusive attempt
- What we can do





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# Light reading

#### Website: fa.its.tudelft.nl/~hart

- K. P. Hart, B. van der Steeg, Span, chainability and the continua ℍ\* and I<sub>u</sub>, Topology and its Applications, 151, 1–3 (2005), 226–237.
- D. Bartošová, K. P. Hart, L. Hoehn, B. van der Steeg, Lelek's problem is not a metric problem, to appear.

