Algebraic Topology, of sorts. Part II

Non impeditus ab ulla scientia

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Galway in Brum, 15 July, 2010: 10:00 – 11:00

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Outline

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Chainability

Definition

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$[0, 1]$ is chainable; the circle S^1 is not.

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$[0, 1]$ has all spans zero, S^1 has all spans non-zero

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The problem

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Question (Lelek)

What about the converse?

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What about the converse?

This is an important problem in metric continuum theory.

But it makes non-metric sense as well.

Implications

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$$\downarrow$$

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\begin{array}{ccc}
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 & & \downarrow \\
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$$\begin{array}{ccc}
\sigma X = 0 & \leftarrow & \frac12\sigma X = 0 \\
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where $\operatorname{Ex} U = \beta\mathbb{H} \setminus \operatorname{cl}(\mathbb{H} \setminus U)$
(the largest open set in $\beta\mathbb{H}$ that intersects $\mathbb{H}$ in U).

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$\mathbb{H}^*$ is not chainable

The open cover $\{O_0, O_1, O_2, O_3\}$ of $\mathbb{H}^*$ does not have a chain refinement — nice exercise, but a bit convoluted.

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The spans of $\mathbb{H}^*$

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and that extension's restriction $f^* : \mathbb{H}^* \to \mathbb{H}^*$.

Its graph witnesses that the surjective span of $\mathbb{H}^*$ is non-zero and hence so are the other three.

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Other candidates

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What can we say about the spans of the $\mathbb{I}_u$?

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The span of $\mathbb{I}_u$

Theorem

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Theorem

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The proof is like that for $\mathbb{H}^*$: the continua $\mathbb{I}_u$ contain subcontinua that are quite similar to $\mathbb{H}^*$ and they allow an analogue of the graph of $x \mapsto x+1$.

The other spans of $\mathbb{I}_u$

Theorem (CH)

The surjective span of $\mathbb{I}_u$ is non-zero.

The other spans of $\mathbb{I}_u$

Theorem (CH)

The surjective span of $\mathbb{I}_u$ is non-zero.

The proof is more involved and can best be illustrated with a picture.

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Ultrapower of a lattice L by an ultrafilter u on a cardinal κ:

$$L^\kappa/{\sim_u}$$

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where $f \sim_u g$ means $\{\alpha : f(\alpha) = g(\alpha)\} \in u$.
L and $L^\kappa/{\sim_u}$ have exactly the same algebraic properties.

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Now then, . . . , $2^{\mathbb{I}}$ satisfies 'chainability' but $L^\kappa/{\sim_u}$ does not, so

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Now then, . . . , $2^{\mathbb{I}}$ satisfies 'chainability' but $L^\kappa/{\sim_u}$ does not, so

unlike the dimensions, chainability is not expressible in first-order terms.

A formula for chainability

The natural formulation is an $L_{\omega_1,\omega}$-formula.

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$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)$$
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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an n-element chain refinement.

It (indeed) suffices to consider four-element open covers only.

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- not a property reducible to bases or
- not first-order.

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Reflection

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Let X be a counterexample, let $L \prec 2^X$ (an elementary sublattice). Then wL is a metrizable counterexample. $\square$

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Not quite . . . because of what we have just seen.

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Theorem

In this situation:

- *wL is chainable iff X is chainable*
- *wL has span zero iff X has span zero (any kind)*

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Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.

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Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.

Furthermore, one needs only consider covers and refinements that belong to a certain base.

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Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$.

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This gives the easy part: if there is a 'bad' continuum in $X \times X$ then there is one in M and it is equally bad in $wL \times wL$.

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For the converse ...

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Span zero, continued

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Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto Z.

Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK}.

Span zero, the real argument

Apply Shelah's Ultrapower theorem

Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal κ, an ultrafilter u on κ and an isomorphism $h : \prod_u (2^{X \times X}) \to \prod_u wK$ (which can be taken to be the identity on K).

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How does that help?

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How does that help?

For that we need some topology.

Dualizing ultrapowers

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- $p_Y : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto y$).

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- $Z_X = p_{X \times X}\big[wh[Z_u]\big]$ is a continuum in $X \times X$.

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- And

$$q_K[Z_X] = q_K\Bigl[p_{X\times X}\bigl[wh[Z_u]\bigr]\Bigr] = p_{wK}\Bigl[(wh)^{-1}\bigl[wh[Z_u]\bigr]\Bigr] = Z$$

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- $Z_X = p_{X \times X}\big[wh[Z_u]\big]$ is a continuum in $X \times X$.
- And

$$q_K[Z_X] = q_K\Big[p_{X\times X}\big[wh[Z_u]\big]\Big] = p_{wK}\Big[(wh)^{-1}\big[wh[Z_u]\big]\Big] = Z$$

So, that's it!?

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Span zero, the real argument

Back to $Z \subseteq wK$.

- Let $Z_u = \mathrm{cl}(\kappa \times Z) \cap p_\kappa^{\leftarrow}(u)$.
- Z_u is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ (wh is dual to h).
- $Z_X = p_{X \times X}\big[wh[Z_u]\big]$ is a continuum in $X \times X$.
- And

$$q_K[Z_X] = q_K\Big[p_{X \times X}\big[wh[Z_u]\big]\Big] = p_{wK}\Big[(wh)^{-1}\big[wh[Z_u]\big]\Big] = Z$$

So, that's it!? Almost.

Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2.

Span zero, the real argument

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Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

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Span zero, the real argument

First expand the language of lattice with two function symbols π_1 and π_2.

Apply Shelah's theorem with this extended language. Then Z_X will inherit the mapping properties that Z has.

Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.

Postscript

Logan Hoehn has constructed a metrizable continuum that is non-chainable but that has span zero.

Outline

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Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart, B. van der Steeg,
Span, chainability and the continua $\mathbb{H}^*$ *and* $\mathbb{I}_u$, Topology and its Applications, 151, 1–3 (2005), 226–237.

D. Bartošová, K. P. Hart, L. Hoehn, B. van der Steeg,
Lelek's problem is not a metric problem, to appear.