ω^*, ω_1^* and non-trivial autohomeomorphisms

Quidquid latine dictum sit, altum videtur

K. P. Hart

Faculteit EWI
TU Delft

Tyler, 19 March 2011: 09:25–10:00

Delft University of Technology

A basic question

All cardinals carry the discrete topology.

Question (M. Turzanski)

Are there different infinite cardinals κ and λ such that κ^* and λ^* are homeomorphic?
Equivalently: are there different infinite cardinals κ and λ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are isomorphic?

Delft University of Technology

First result

Theorem (Frankiewicz 1978)

$\mathfrak{b} = \mathfrak{c}$ *implies* ω^* *and* κ^* *are not homeomorphic whenever* κ *is regular uncountable.*

Proof.

Slightly convoluted proof involving P-points and (non-)measurable cardinals. □

Second result

Theorem (Frankiewicz 1977)

If there are different κ and λ such that κ^* and λ^* are homeomorphic then ω^* and ω_1^* are also homeomorphic.

Proof.

Let κ be the smallest (if any) cardinal for which there is λ different from κ such that κ^* and λ^* are homeomorphic.
If $h : \lambda^* \to \kappa^*$ is a homeomorphism then $h[(\kappa^+)^*] = A^*$ for some $A \subseteq \kappa$.
Minimality: $|A| = \kappa$, hence κ^* and $(\kappa^+)^*$ are homeomorphic. □

Delft University of Technology

More about the proof

Proof (continued).

If κ were singular we could prove κ^* and $(\kappa^+)^*$ are not homeomorphic.
If κ were regular uncountable then we could prove κ^* and $(\kappa^+)^*$ are not homeomorphic.
So $\kappa = \omega$ and we find that ω^* and ω_1^* are homeomorphic under the assumption that there is such a κ. □

Delft University of Technology

Third result

Theorem (Balcar and Frankiewicz 1978)

ω_1^* and ω_2^* are not homeomorphic.

Proof.

If ω_1^* and ω_2^* are homeomorphic then so are ω^* and ω_1^*.
But, in that case we could infer that ${}^{\omega}\omega$ contains both an ω_1- and an ω_2-scale, which would then imply $\omega_1 = \omega_2$.
(See later for a proof of the scale assertion.) □

Delft University of Technology

Consequence

Corollary

If $\omega_1 \leqslant \kappa < \lambda$ *then* κ^* *and* λ^* *are not homeomorphic.*

So we are left with

Question

Are ω^* and ω_1^* ever homeomorphic?

Consequences of 'yes'

Easiest consequence: $2^{\aleph_0} = 2^{\aleph_1}$;

those are the respective weights of ω^* and ω_1^*

(or cardinalities of $\mathcal{P}(\omega)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$).

So CH implies 'no'.

Delft University of Technology

Consequences of 'yes'

We consider the set $\omega \times \omega_1$.

We put

- $V_n = \{n\} \times \omega_1$
- $H_\alpha = \omega \times \{\alpha\}$
- $B_\alpha = \omega \times \alpha$
- $E_\alpha = \omega \times [\alpha, \omega_1)$

Let $\gamma : (\omega \times \omega_1)^* \to (\omega \times \omega)^*$ be a homeomorphism.
We can assume $\gamma[V_n^*] = \big(\{n\} \times \omega\big)^*$ for all n.

Delft University of Technology

An ω_1-scale

Choose e_α, for each α, such that $\gamma[E^*_\alpha] = e^*_\alpha$.
Define $f_\alpha : \omega \to \omega$ by

$$f_\alpha(m) = \min\{n : \langle m, n\rangle \in e_\alpha\}$$

Verify that $f_\alpha \leqslant^* f_\beta$ when $\alpha < \beta$.
If $f : \omega \to \omega$ then $e_\alpha \cap f$ is finite for many α and for those α we have $f < f_\alpha$.
So $\langle f_\alpha : \alpha < \omega_1\rangle$ is an ω_1-scale.

And so $\mathrm{MA} + \neg\mathrm{CH}$ implies 'no'.

TU Delft
Delft University of Technology

A strong Q-sequence

Choose h_α, for each α, such that $\gamma[H_\alpha^*] = h_\alpha^*$.

$\{h_\alpha : \alpha < \omega_1\}$ is an almost disjoint family. And a very special one at that.

Given $x_\alpha \subseteq h_\alpha$ for each α there is x such that $x \cap h_\alpha =^* x_\alpha$ for all α.

Basically $x^* = h[X^*]$, where X is such that $(X \cap H_\alpha)^* = \gamma^\leftarrow[x_\alpha^*]$ for all α.

Such *strong Q-sequences* exist consistently (Steprāns).

Delft University of Technology

Even better (or worse?)

It is consistent to have

- $\mathfrak{d} = \omega_1$
- a strong Q-sequence
- $2^{\aleph_0} = 2^{\aleph_1}$

simultaneously (Chodounsky).
(Actually second implies third.)

TUDelft
Delft University of Technology

An autohomeomorphism of ω_1^*

Work with the set $D = \mathbb{Z} \times \omega_1$ — so now $\gamma : D^* \to \omega^*$.

Define $\Sigma : D \to D$ by $\Sigma(n, \alpha) = \langle n + 1, \alpha \rangle$.

Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of ω^*.

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \to b$ between cofinite sets such that $\tau[x^*] = \sigma[x \cap a]^*$ for all subsets x of ω

Delft University of Technology

How does that work?

- $\{H_\alpha^* : \alpha < \omega_1\}$ is a *maximal* disjoint family of Σ^*-invariant clopen sets.
- $\Sigma^*[V_n^*] = V_{n+1}^*$ for all n
- if $V_n^* \subseteq C^*$ for all n then $E_\alpha \subseteq C$ for some α and hence $H_\alpha^* \subseteq C^*$ for all but countably many α.

How does that work?

In ω we have sets h_α, v_n, b_α and e_α that mirror this:

- $\{h^*_\alpha : \alpha < \omega_1\}$ us a *maximal* disjoint family of τ-invariant clopen sets.
- $\tau[v^*_n] = v^*_{n+1}$ for all n
- if $v^*_n \subseteq c^*$ for all n then $e^*_\alpha \subseteq c^*$ for some α and hence $h^*_\alpha \subseteq c^*$ for all but countably many α.

Delft University of Technology

How does that work?

The assumption that $\tau = \sigma^*$ for some σ leads, via some bookkeeping, to a set c with the properties that

- $v_n \subseteq^* c$ for all n and
- $h_\alpha \not\subseteq^* c$ for uncountably many α (in fact all but countably many).

which neatly contradicts what's on the previous slide ...

Some more details

Assume we have a $\sigma : a \to b$ inducing the isomorphism (without loss of generality $a = \omega$).

Split ω into I and F — the unions of the Infinite and Finite orbits, respectively.

An infinite orbit must meet an h_α in an infinite set — and at most two of these.

Why is 'two' even possible?
If the orbit of n is two-sided infinite then both $\{\sigma^k(n) : k \leqslant 0\}^*$ and $\{\sigma^k(n) : k \geqslant 0\}^*$ are τ-invariant.

Delft University of Technology

Some more details

It follows that $h_\alpha \subseteq^* F$ for all but countably many α. and hence $v_n \cap F$ is infinite for all n.

- each $h_\alpha \cap F$ is a union of finite orbits
- those finite orbits have arbitrarily large cardinality
  better still, the cardinalities convere to ω.
- Our set c is the union of I and half of each finite orbit.

Certainly $h_\alpha \setminus c$ is infinite for our co-countably many α.

Final detail

- Write each finite orbit as $\{\sigma^k(n) : -l \leqslant k \leqslant m\}$
- with $n \in v_0$ and $|m - l| \leqslant 1$
- use $\{\sigma^k(n) : -l/2 \leqslant k \leqslant m/2\}$ as a constituent of c.

Comments/frustrations

This feels tantalisingly close to a proof that ω^* and ω_1^* are not homeomorphic, to me anyway, because.

- it seems that τ should be trivial on all (but countably many) h_α; reason: h_α should be (the graph of) a function and so τ should be induced by the shift on h_α
- an argument with complete accumulation points should then give enough triviality to make those shifts cohere
- which would then lead to a contradiction

None of which I have been able to prove . . .

TUDelft
Delft University of Technology

More Info

Website: `http://fa.its.tudelft.nl/~hart`

Delft University of Technology