ω^*, ω_1^* and non-trivial autohomeomorphisms

Quidquid latine dictum sit, altum videtur

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Outline

A basic question

All cardinals carry the discrete topology.

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Question (M. Turzanski)

Are there different infinite cardinals κ and λ such that κ^* and λ^* are homeomorphic?

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All cardinals carry the discrete topology.

Question (M. Turzanski)

Are there different infinite cardinals κ and λ such that κ^* and λ^* are homeomorphic?
Equivalently: are there different infinite cardinals κ and λ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are isomorphic?

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First result

Theorem (Frankiewicz 1978)

$\mathfrak{b} = \mathfrak{c}$ *implies* ω^* *and* κ^* *are not homeomorphic whenever* κ *is regular uncountable.*

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Proof.

Slightly convoluted proof involving P-points and (non-)measurable cardinals. □

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Second result

Theorem (Frankiewicz 1977)

If there are different κ and λ such that κ^* and λ^* are homeomorphic then ω^* and ω_1^* are also homeomorphic.

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Let κ be the smallest (if any) cardinal for which there is λ different from κ such that κ^* and λ^* are homeomorphic.

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Let κ be the smallest (if any) cardinal for which there is λ different from κ such that κ^* and λ^* are homeomorphic.
If $h : \lambda^* \to \kappa^*$ is a homeomorphism then $h[(\kappa^+)^*] = A^*$ for some $A \subseteq \kappa$

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If $h : \lambda^* \to \kappa^*$ is a homeomorphism then $h[(\kappa^+)^*] = A^*$ for some $A \subseteq \kappa$.
Minimality: $|A| = \kappa$, hence κ^* and $(\kappa^+)^*$ are homeomorphic. $\square$

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More about the proof

Proof (continued).

If κ were singular we could prove κ^* and $(\kappa^+)^*$ are not homeomorphic.

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If κ were regular uncountable then we could prove κ^* and $(\kappa^+)^*$ are not homeomorphic.

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More about the proof

Proof (continued).

If κ were singular we could prove κ^* and $(\kappa^+)^*$ are not homeomorphic.
If κ were regular uncountable then we could prove κ^* and $(\kappa^+)^*$ are not homeomorphic.
So $\kappa = \omega$ and we find that ω^* and ω_1^* are homeomorphic under the assumption that there is such a κ. $\square$

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Third result

Theorem (Balcar and Frankiewicz 1978)

ω_1^* and ω_2^* are not homeomorphic.

Proof.

If ω_1^* and ω_2^* are homeomorphic then so are ω^* and ω_1^*.

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If ω_1^* and ω_2^* are homeomorphic then so are ω^* and ω_1^*.
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(See later for a proof of the scale assertion.) $\square$

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Corollary

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So we are left with

Question

Are ω^* and ω_1^* ever homeomorphic?

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Outline

1. History
2. More recent history
3. A non-trivial autohomeomorphism

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Consequences of 'yes'

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So CH implies ‘no’.

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Let $\gamma : (\omega \times \omega_1)^* \to (\omega \times \omega)^*$ be a homeomorphism.
We can assume $\gamma[V_n^*] = \bigl(\{n\} \times \omega\bigr)^*$ for all n.

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And so $\mathrm{MA} + \neg\mathrm{CH}$ implies 'no'.

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Basically $x^* = h[X^*]$, where X is such that $(X \cap H_\alpha)^* = \gamma^{\leftarrow}[x_\alpha^*]$ for all α.

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Such *strong Q-sequences* exist consistently (Steprāns).

Even better (or worse?)

It is consistent to have

- $\mathfrak{d} = \omega_1$
- a strong Q-sequence
- $2^{\aleph_0} = 2^{\aleph_1}$

simultaneously (Chodounsky).

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(Actually second implies third.)

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An autohomeomorphism of ω_1^*

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Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of ω^*.

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Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of ω^*.

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \to b$ between cofinite sets such that $\tau[x^*] = \sigma[x \cap a]^*$ for all subsets x of ω

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- if $V^*_n \subseteq C^*$ for all n then $E_\alpha \subseteq C$ for some α and hence $H^*_\alpha \subseteq C^*$ for all but countably many α.

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which neatly contradicts what's on the previous slide ...

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Some more details

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Split ω into I and F — the unions of the Infinite and Finite orbits, respectively.

An infinite orbit must meet an h_α in an infinite set — and at most two of these.

Why is 'two' even possible?
If the orbit of n is two-sided infinite then both $\{\sigma^k(n) : k \leqslant 0\}^*$ and $\{\sigma^k(n) : k \geqslant 0\}^*$ are τ-invariant.

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It follows that $h_\alpha \subseteq^* F$ for all but countably many α. and hence $v_n \cap F$ is infinite for all n.

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- Our set c is the union of I and half of each finite orbit.

Certainly $h_\alpha \setminus c$ is infinite for our co-countably many α.

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- with $n \in v_0$ and $|m - l| \leqslant 1$
- use $\{\sigma^k(n) : -l/2 \leqslant k \leqslant m/2\}$ as a constituent of c.

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None of which I have been able to prove ...

More Info

Website: `http://fa.its.tudelft.nl/~hart`

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