ω^*, ω_1^* and non-trivial autohomeomorphisms

Quidquid latine dictum sit, altum videtur

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A basic question

All cardinals carry the discrete topology.

Question (The Katowice Problem)

Are there different infinite cardinals κ and λ such that κ^* and λ^* are homeomorphic?
Equivalently: are there different infinite cardinals κ and λ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are isomorphic?

Gut reaction

Of course not! That would be shocking.

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A word of warning

Remember: people thought that $\kappa < \lambda$ would imply $2^\kappa < 2^\lambda$.

And had a hard time proving it.

And then they learned that it was unprovable.

Very unprovable: one can specify regular cardinals at will and create a model in which their 2-powers are equal.

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A glimmer of hope

If $\kappa < \lambda$ then $\beta\kappa$ and $\beta\lambda$ are not homeomorphic
(or $\mathcal{P}(\kappa)$ and $\mathcal{P}(\lambda)$ are not isomorphic)
even if $2^\kappa = 2^\lambda$.

Their sets of isolated points (atoms) have different cardinalities.

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More hope

Theorem (Frankiewicz 1977)

The minimum cardinal κ (if any) such that κ^* is homeomorphic to λ^* for some $\lambda > \kappa$ must be ω.

Theorem (Balcar and Frankiewicz 1978)

ω_1^* and ω_2^* are not homeomorphic.

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Our gut was not completely wrong

Corollary

If $\omega_1 \leqslant \kappa < \lambda$ then κ^* and λ^* are not homeomorphic, and if $\omega_2 \leqslant \lambda$ then ω^* and λ^* are not homeomorphic.

So we are left with

Question

Are ω^* and ω_1^* ever homeomorphic?

Consequences of ‘yes’

Easiest consequence: $2^{\aleph_0} = 2^{\aleph_1}$;

those are the respective weights of ω^* and ω_1^*

(or cardinalities of $\mathcal{P}(\omega)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$).

So CH implies ‘no’.

Consequences of 'yes'

We consider the set $\omega \times \omega_1$.

We put

- $V_n = \{n\} \times \omega_1$
- $H_\alpha = \omega \times \{\alpha\}$
- $B_\alpha = \omega \times \alpha$
- $E_\alpha = \omega \times [\alpha, \omega_1)$

Let $\gamma : (\omega \times \omega_1)^* \to (\omega \times \omega)^*$ be a homeomorphism.
We can assume $\gamma[V_n^*] = \bigl(\{n\} \times \omega\bigr)^*$ for all n.

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An ω_1-scale

Choose e_α, for each α, such that $\gamma[E_\alpha^*] = e_\alpha^*$.
Define $f_\alpha : \omega \to \omega$ by

$$f_\alpha(m) = \min\{n : \langle m, n\rangle \in e_\alpha\}$$

Verify that $f_\alpha \leqslant^* f_\beta$ when $\alpha < \beta$.
If $f : \omega \to \omega$ then $e_\alpha \cap f$ is finite for many α and for those α we have $f <^* f_\alpha$.
So $\langle f_\alpha : \alpha < \omega_1\rangle$ is an ω_1-scale.

And so $\mathrm{MA} + \neg\mathrm{CH}$ implies 'no'.

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A strong Q-sequence

Choose h_α, for each α, such that $\gamma[H_\alpha^*] = h_\alpha^*$.

$\{h_\alpha : \alpha < \omega_1\}$ is an almost disjoint family.
And a very special one at that.

Given $x_\alpha \subseteq h_\alpha$ for each α there is x such that $x \cap h_\alpha =^* x_\alpha$ for all α.

Basically $x^* = h[X^*]$, where X is such that $(X \cap H_\alpha)^* = \gamma^{\leftarrow}[x_\alpha^*]$ for all α.

Such *strong Q-sequences* exist consistently (Steprāns).

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Even better (or worse?)

It is consistent to have

- $\mathfrak{d} = \omega_1$
- a strong Q-sequence
- $2^{\aleph_0} = 2^{\aleph_1}$

simultaneously (Chodounsky).

(Actually second implies third.)

An autohomeomorphism of ω_1^*

Work with the set $D = \mathbb{Z} \times \omega_1$ — so now $\gamma : D^* \to \omega^*$.

Define $\Sigma : D \to D$ by $\Sigma(n, \alpha) = \langle n+1, \alpha \rangle$.

Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of ω^*.

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \to b$ between cofinite sets such that $\tau[x^*] = \sigma[x \cap a]^*$ for all subsets x of ω

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How does that work?

- $\{H^*_\alpha : \alpha < \omega_1\}$ is a *maximal* disjoint family of Σ^*-invariant clopen sets.
- $\Sigma^*[V^*_n] = V^*_{n+1}$ for all n
- if $V^*_n \subseteq C^*$ for all n then $E_\alpha \subseteq C$ for some α and hence $H^*_\alpha \subseteq C^*$ for all but countably many α.

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How does that work?

In ω we have sets h_α, v_n, b_α and e_α that mirror this:

- $\{h^*_\alpha : \alpha < \omega_1\}$ is a *maximal* disjoint family of τ-invariant clopen sets.
- $\tau[v^*_n] = v^*_{n+1}$ for all n
- if $v^*_n \subseteq c^*$ for all n then $e^*_\alpha \subseteq c^*$ for some α and hence $h^*_\alpha \subseteq c^*$ for all but countably many α.

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How does that work?

The assumption that $\tau = \sigma^*$ for some σ leads, via some bookkeeping, to a set c with the properties that

- $v_n \subseteq^* c$ for all n and
- $h_\alpha \nsubseteq^* c$ for uncountably many α (in fact all but countably many).

which neatly contradicts what's on the previous slide ...

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Some more details

Assume we have a $\sigma : a \to b$ inducing the isomorphism (without loss of generality $a = \omega$).

Split ω into I and F — the unions of the Infinite and Finite orbits, respectively.

An infinite orbit must meet an h_α in an infinite set — and at most two of these.

Why is 'two' even possible?
If the orbit of n is two-sided infinite then both $\{\sigma^k(n) : k \leqslant 0\}^*$ and $\{\sigma^k(n) : k \geqslant 0\}^*$ are τ-invariant.

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Some more details

It follows that $h_\alpha \subseteq^* F$ for all but countably many α and hence $v_n \cap F$ is infinite for all n.

- each $h_\alpha \cap F$ is a union of finite orbits
- those finite orbits have arbitrarily large cardinality
  better still, the cardinalities converge to ω.
- Our set c is the union of I and half of each finite orbit.

Certainly $h_\alpha \setminus c$ is infinite for our co-countably many α.

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Finer detail

- Write each finite orbit as $\{\sigma^k(n) : -l \leqslant k \leqslant m\}$
- with $n \in v_0$ and $|m - l| \leqslant 1$
- use $\{\sigma^k(n) : -l/2 \leqslant k \leqslant m/2\}$ as a constituent of c.

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Comment

This tells us that a homeomorphism between ω^* and ω_1^* must be quite complicated.
At least as complicated as a non-trivial autohomeomorphism.

Veličković' analysis of autohomeomorphisms of ω^* shows that the ones that are remotely describable (by Borel maps say) are trivial.
Our Σ is very trivial, so it's the putative homeomorphism γ that is badly describable.

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Frustration

This feels tantalisingly close to a proof that ω^* and ω_1^* are not homeomorphic, to me anyway, because.

- it seems that τ should be trivial on all (but countably many) h_α; reason: h_α should be (the graph of) a function and so τ should be induced by the shift on h_α
- an argument with complete accumulation points should then give enough triviality to make those shifts cohere
- which should then lead to a contradiction

None of which I have been able to prove ...

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More Info

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