

ω^* , ω_1^* and non-trivial autohomeomorphisms

Quidquid latine dictum sit, altum videtur

K. P. Hart

Fakulta EMK
TU Delft

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Outline

- 1 History
- 2 Working toward $0 = 1$
- 3 A non-trivial autohomeomorphism

A basic question

All cardinals carry the discrete topology.

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Question (The Katowice Problem)

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Gut reaction

Of course not!

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Gut reaction

Of course not! That would be shocking.

A word of warning

Remember: people thought that $\kappa < \lambda$ would imply $2^\kappa < 2^\lambda$.

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Very unprovable: one can specify regular cardinals at will and create a model in which their 2-powers are equal.

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Their sets of isolated points (atoms) have different cardinalities.

More hope

Theorem (Frankiewicz 1977)

The minimum cardinal κ (if any) such that κ^ is homeomorphic to λ^* for some $\lambda > \kappa$ must be ω .*

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Theorem (Balcar and Frankiewicz 1978)

ω_1^ and ω_2^* are not homeomorphic.*

Our gut was not completely wrong

Corollary

If $\omega_1 \leq \kappa < \lambda$ then κ^ and λ^* are not homeomorphic*

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If $\omega_1 \leq \kappa < \lambda$ then κ^ and λ^* are not homeomorphic, and if $\omega_2 \leq \lambda$ then ω^* and λ^* are not homeomorphic.*

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Corollary

If $\omega_1 \leq \kappa < \lambda$ then κ^ and λ^* are not homeomorphic, and if $\omega_2 \leq \lambda$ then ω^* and λ^* are not homeomorphic.*

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Question

Are ω^* and ω_1^* ever homeomorphic?

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So CH implies 'no'.

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We can assume $\gamma[V_n^*] = (\{n\} \times \omega)^*$ for all n .

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And so $\text{MA} + \neg\text{CH}$ implies 'no'.

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Such *strong Q -sequences* exist consistently (Steprāns).

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(Actually second implies third.)

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Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of ω^* .

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \rightarrow b$ between cofinite sets such that $\tau[x^*] = \sigma[x \cap a]^*$ for all subsets x of ω

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- if $V_n^* \subseteq C^*$ for all n then $E_\alpha \subseteq C$ for some α and hence $H_\alpha^* \subseteq C^*$ for all but countably many α .

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which neatly contradicts what's on the previous slide . . .

Some more details

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Why is ‘two’ even possible?

If the orbit of n is two-sided infinite then both $\{\sigma^k(n) : k \leq 0\}^*$ and $\{\sigma^k(n) : k \geq 0\}^*$ are τ -invariant.

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Certainly $h_\alpha \setminus c$ is infinite for our co-countably many α .

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- with $n \in v_0$ and $|m - l| \leq 1$
- use $\{\sigma^k(n) : -l/2 \leq k \leq m/2\}$ as a constituent of c .

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Veličković' analysis of autohomeomorphisms of ω^* shows that the ones that are remotely describable (by Borel maps say) are trivial. Our Σ is very trivial, so it's the putative homeomorphism γ that is badly describable.

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None of which I have been able to prove . . .

More Info

Website: `http://fa.its.tudelft.nl/~hart`

