ω^* and ω_1^*

Quidquid latine dictum sit, altum videtur

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A basic question

All cardinals carry the discrete topology.

Question (Marian Turzanski)

Are ω^* and ω_1^* homeomorphic?
Equivalently: are the Boolean algebras $\mathcal{P}(\omega)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are isomorphic?

He asked this when he was a graduate student after he was assigned Parovičenko's paper "A universal continuum of weight $\aleph$".

A more general question

Question

Are there different infinite cardinals κ and λ such that κ^* and λ^* are homeomorphic?
Equivalently: are there different infinite cardinals κ and λ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are isomorphic?

It turns out that Turzanski's question forms the only interesting case of the general question.

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What we are talking about: topologically

We take the Čech-Stone compactification, $\beta\kappa$, of the discrete space κ.
Characterizing properties of $\beta\kappa$:

- it is compact Hausdorff
- κ is a dense subset
- for every $A \subseteq \kappa$ the closures of A and $\kappa \setminus A$ in $\beta\kappa$ are disjoint

κ^* is $\beta\kappa \setminus \kappa$
(generally we write $A^* = \overline{A} \setminus A$ for $A \subseteq \kappa$)

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What we are talking about: algebraically

Consider the power set, $\mathcal{P}(\kappa)$, of κ.

It is a Boolean algebra, with operations $\cup$, $\cap$ and $\kappa \setminus \cdot$

The family *fin*, of finite sets, is an ideal in this algebra.

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What we are talking about: the connection

Stone duality connects these two types of structures.

The family of clopen subsets of $\beta\kappa$ is $\{\overline{A} : A \in \mathcal{P}(\kappa)\}$, which, by the characterizing properties, is isomorphic to $\mathcal{P}(\kappa)$.

The family of clopen subsets of κ is $\{A^* : A \in \mathcal{P}(\kappa)\}$, which, by the characterizing properties, is isomorphic to $\mathcal{P}(\kappa)/\mathit{fin}$.
For observe: $A^* = B^*$ iff A and B differ by a finite set.

Two results

Theorem (Frankiewicz 1977)

The minimum cardinal κ (if any) such that κ^* is homeomorphic to λ^* for some $\lambda > \kappa$ must be ω.

Theorem (Balcar and Frankiewicz 1978)

ω_1^* and ω_2^* are not homeomorphic.

Assume there are κ and λ ...

Let κ be minimal such that there is $\lambda > \kappa$ for which κ^* and λ^* are homeomorphic.

Proposition

If $\kappa < \mu < \lambda$ then κ^* and μ^* are homeomorphic.

Proof.

Let $h : \lambda^* \to \kappa^*$ be a homeomorphism and take $A \subseteq \kappa$ such that $A^* = h[\mu^*]$.
Note: $|A| < \mu$, so by minimality of κ we must have $|A| = \kappa$. □

Assume κ is the minimal ...

Proposition

$\kappa = \omega$

Proof.

Let $h : \kappa^* \to (\kappa^+)^*$ be a homeomorphism.
For $\alpha < \kappa$ take $A_\alpha \subseteq \kappa^+$ such that $A_\alpha^* = h[\alpha^*]$ and let $A = \bigcup_{\alpha<\kappa} A_\alpha$.
Note: $|A_\alpha| = |\alpha| < \kappa$ for all α, by minimality of κ, so $|A| \leqslant \kappa$. $\square$

Assume κ is the minimal ...

Proposition

$\kappa = \omega$

Proof, continued.

Take $B \subseteq \kappa$ such that $A^* = h[B^*]$, and so $(\kappa^+ \setminus A)^* = h[(\kappa \setminus B)^*]$.
This implies $|\kappa \setminus B| = \kappa$.

But $\alpha^* \subseteq B^*$, which means $\alpha \setminus B$ is finite, for all α.
And so $|\kappa \setminus B| \leqslant \omega$. $\square$

Scales

Let $\kappa > \omega$ and assume ω^* and κ^* are homeomorphic.
Consider $\omega \times \kappa$ instead of κ and let $\gamma : (\omega \times \kappa)^* \to \omega^*$ be a homeomorphism.
Let $V_n = \{n\} \times \kappa$ and choose $v_n \subseteq \omega$ such that $v_n^* = h[V_n^*]$.
We may rearrange the v_n to make them disjoint and even assume $v_n = \{n\} \times \omega$ for all n.

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Scales

For $\alpha < \kappa$ let $E_\alpha = \omega \times [\alpha, \kappa)$ and take $e_\alpha \subseteq \omega \times \omega$ such that $e_\alpha^* = h[E_\alpha^*]$.
Define $f_\alpha : \omega \to \omega$ by

$$f_\alpha(n) = \min\{k : \{n\} \times [k, \omega) \subseteq e_\alpha\}$$

Note: $f_\alpha \leqslant^* f_\beta$ if $\alpha < \beta$, i.e., $\{n : f_\alpha(n) > f_\beta(n)\}$ is finite.
For every $f : \omega \to \omega$ there is an α such that $f \leqslant^* f_\alpha$.
$\langle f_\alpha : \alpha < \kappa \rangle$ is a κ-scale.

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Scales

Assume ω_1^* and ω_2^* are homeomorphic.

Then ω^* and ω_1^* must also be homeomorphic.

But then we'd have an ω_1-scale and an ω_2-scale and hence a contradiction.

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Consequences

Corollary

If $\omega_1 \leqslant \kappa < \lambda$ then κ^* and λ^* are not homeomorphic, and if $\omega_2 \leqslant \lambda$ then ω^* and λ^* are not homeomorphic.

So we are left with

Question

Are ω^* and ω_1^* ever homeomorphic?

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So, what if they are homeomorphic?

Easiest consequence: $2^{\aleph_0} = 2^{\aleph_1}$;

those are the respective weights of ω^* and ω_1^*

(or cardinalities of $\mathcal{P}(\omega)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$).

So CH implies ‘no’.

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An ω_1-scale

Using the scales we get

$$\mathfrak{d} = \omega_1$$

And so $\mathrm{MA} + \neg\mathrm{CH}$ implies ‘no’.

A strong Q-sequence

In $\omega \times \omega_1$ let $H_\alpha = \omega \times \{\alpha\}$ and, for each α, choose $h_\alpha \subseteq \omega \times \omega$ such that $\gamma[H^*_\alpha] = h^*_\alpha$.

$\{h_\alpha : \alpha < \omega_1\}$ is an almost disjoint family.
And a very special one at that.

Given $x_\alpha \subseteq h_\alpha$ for each α there is x such that $x \cap h_\alpha =^* x_\alpha$ for all α.

Basically $x^* = h[X^*]$, where X is such that $(X \cap H_\alpha)^* = \gamma^{\leftarrow}[x^*_\alpha]$ for all α.

Such *strong Q-sequences* exist consistently (Steprāns).

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Even better (or worse?)

It is consistent to have

- $\mathfrak{d} = \omega_1$
- a strong Q-sequence
- $2^{\aleph_0} = 2^{\aleph_1}$

simultaneously (Chodounsky).

(Actually second implies third.)

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An autohomeomorphism of ω_1^*

Work with the set $D = \mathbb{Z} \times \omega_1$ — so now $\gamma : D^* \to \omega^*$.

Define $\Sigma : D \to D$ by $\Sigma(n, \alpha) = \langle n + 1, \alpha \rangle$.

Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of ω^*.

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \to b$ between cofinite sets such that $\tau[x^*] = \sigma[x \cap a]^*$ for all subsets x of ω

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How does that work?

- $\{H_\alpha^* : \alpha < \omega_1\}$ is a *maximal* disjoint family of Σ^*-invariant clopen sets.
- $\Sigma^*[V_n^*] = V_{n+1}^*$ for all n
- if $V_n^* \subseteq C^*$ for all n then $E_\alpha \subseteq C$ for some α and hence $H_\alpha^* \subseteq C^*$ for all but countably many α.

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How does that work?

In ω we have sets h_α, v_n, b_α and e_α that mirror this:

- $\{h_\alpha^* : \alpha < \omega_1\}$ is a *maximal* disjoint family of τ-invariant clopen sets.
- $\tau[v_n^*] = v_{n+1}^*$ for all n
- if $v_n^* \subseteq c^*$ for all n then $e_\alpha^* \subseteq c^*$ for some α and hence $h_\alpha^* \subseteq c^*$ for all but countably many α.

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How does that work?

The assumption that $\tau = \sigma^*$ for some σ leads, via some bookkeeping, to a set c with the properties that

- $v_n \subseteq^* c$ for all n and
- $h_\alpha \not\subseteq^* c$ for uncountably many α (in fact all but countably many).

which neatly contradicts what's on the previous slide ...

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Some more details

Assume we have a $\sigma : a \to b$ inducing the isomorphism (without loss of generality $a = \omega$).

Split ω into I and F — the unions of the Infinite and Finite orbits, respectively.

An infinite orbit must meet an h_α in an infinite set — and at most two of these.

Why is ‘two’ even possible?
If the orbit of n is two-sided infinite then both $\{\sigma^k(n) : k \leqslant 0\}^*$ and $\{\sigma^k(n) : k \geqslant 0\}^*$ are τ-invariant.

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Some more details

It follows that $h_\alpha \subseteq^* F$ for all but countably many α and hence $v_n \cap F$ is infinite for all n.

- each $h_\alpha \cap F$ is a union of finite orbits
- those finite orbits have arbitrarily large cardinality
  better still, the cardinalities converge to ω.
- Our set c is the union of I and half of each finite orbit.

Certainly $h_\alpha \setminus c$ is infinite for our co-countably many α.

Finer detail

- Write each finite orbit as $\{\sigma^k(n) : -l \leqslant k \leqslant m\}$
- with $n \in v_0$ and $|m - l| \leqslant 1$
- use $\{\sigma^k(n) : -l/2 \leqslant k \leqslant m/2\}$ as a constituent of c.

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Moral of the story

Don't be afraid to ask questions.

You may be asking the next very interesting one.

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