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ČECH-STONE COMPACTIFICATION

X COMPLETELY REGULAR

THERE IS A COMPACT HAUSDORFF SPACE βX SUCH THAT
- X IS A DENSE SUBSPACE OF βX
- EVERY CONTINUOUS $f : X \to [0,1]$ HAS A CONTINUOUS EXTENSION $\beta f : \beta X \to [0,1]$

THE SECOND PROPERTY CHARACTERIZES βX AMONG ALL <u>COMPACTIFICATIONS</u> OF X.

OTHER CHARACTERIZATIONS

- A AND B DISJOINT ZERO-SETS IN $X \to cl_\beta A \cap cl_\beta B = \emptyset$

■ (X NORMAL) A AND B DISJOINT AND CLOSED IN $X \to cl_\beta A \cap cl_\beta B = \emptyset$.

CONSTRUCTION (E. ČECH)
TAKE EMBEDDING $e : X \longrightarrow [0,1]^{C(X,[0,1])}$ DEFINED BY
$e(x) = (f(x))_{f \in C(X,[0,1])}$.
THEN βX IS THE CLOSURE OF $e[X]$
EXTENSION PROPERTY: $\pi_f \restriction \beta X$ EXTENDS f.

WE CONCENTRATE ON $\beta \mathbb{N}$ AND $\beta \mathbb{R}$.
THESE ARE NORMAL SO WE USE CHARACTERIZATION ■

FIRST $\beta\mathbb{N}$.
FOR $A \subseteq \mathbb{N}$ WRITE $\overline{A} = cl_\beta A$
- $\overline{A}$ IS <u>ALWAYS</u> <u>CLOSED-AND-OPEN</u> (CLOPEN):
  - $\overline{A} \cup \overline{\mathbb{N}\setminus A} = \overline{\mathbb{N}} = \beta\mathbb{N}$
  - $\overline{A} \cap \overline{\mathbb{N}\setminus A} = \emptyset$ (BY ■, AS $A \cap (\mathbb{N}\setminus A) = \emptyset$)
- THE FAMILY $\{\overline{A} : A \subseteq \mathbb{N}\}$ IS A BASE FOR $\beta\mathbb{N}$:
  IF $x \in O$ TAKE V OPEN SUCH THAT
  $$x \in V \subseteq cl_\beta V \subseteq O$$
  NOW: $cl_\beta V = cl_\beta (V \cap \mathbb{N}) = \overline{V \cap \mathbb{N}}$

THIS SHOWS: $\beta\mathbb{N}$ IS EXTREMALLY DISCONNECTED:
IF U AND V ARE DISJOINT AND OPEN THEN $cl_\beta U \cap cl_\beta V = \emptyset$
FOR $cl_\beta U \cap cl_\beta V = \overline{U \cap \mathbb{N}} \cap \overline{V \cap \mathbb{N}} = \emptyset$

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LET $x \in \beta\mathbb{N}$ AND PUT $u_x = \{A \subseteq \mathbb{N} : x \in \overline{A}\}$

- $\emptyset \notin u_x$ AND $\mathbb{N} \in u_x$
- $A \in u_x \wedge B \supseteq A \rightarrow B \in u_x$
- $A, B \in u_x \longrightarrow A \cap B \in u_x$
  - $\overline{A} = \overline{A \setminus B} \cup \overline{A \cap B}$
  - $\overline{A \setminus B} \cap \overline{B} = \emptyset$ (BECAUSE $A \setminus B \cap B = \emptyset$)
  - SO $x \notin \overline{A \setminus B}$
  - BUT $x \in \overline{A}$ SO $x \in \overline{A \cap B}$.
- $A \subseteq \mathbb{N} \rightarrow A \in u_x$ OR $\mathbb{N} \setminus A \in u_x$

$\therefore$ u_x IS AN ULTRAFILTER

CONVERSELY

u AN ULTRAFILTER $\rightarrow \bigcap\{\overline{A} : A \in u\} = \{x_u\}$

- CERTAINLY $\bigcap\{\overline{A} : A \in u\} \neq \emptyset$ COMPACTNESS
- IF $x, y \in \bigcap\{\overline{A} : A \in u\}$ THEN $u \subseteq u_x, u_y$
  BUT u IS AN ULTRAFILTER SO $u = u_x = u_y$
- $x \neq y \rightarrow u_x \neq u_y$
  THERE ARE $A \in u_x$ AND $B \in u_y$ WITH $\overline{A} \cap \overline{B} = \emptyset$
  THEN $A \notin u_y$ AND $B \notin u_y$

THUS $\beta\mathbb{N}$ IS THE SET OF ULTRAFILTERS ON $\mathbb{N}$
AND A BASE FOR THE TOPOLOGY IS
$\{\overline{A} : A \subseteq \mathbb{N}\}$
WHERE $\overline{A} = \{u \in \beta\mathbb{N} : A \in u\}$

ALTERNATIVE DESCRIPTION [TYCHONOFF].

- FOR $x \in (0,1)$ LET $\langle a_i(x) : i \in \mathbb{N}\rangle$ BE THE SEQUENCE OF DIGITS IN ITS BINARY EXPANSION (FAVOURING THE ONE ENDING IN ZEROS).
- THEN $\{a_i : i \in \mathbb{N}\} \subseteq [0,1]^{(0,1)}$
  - LET $A \subseteq \mathbb{N}$ HAVE INFINITE COMPLEMENT AND LET $x = \sum_{i \in A} 2^{-i}$.
    THEN $i \in A \iff a_i(x) = 1$
    SO $cl\{a_i : i \in A\} \cap cl\{a_i : i \notin A\} \subseteq \pi_x^{\leftarrow}(1) \cap \pi_x^{\leftarrow}(0) = \emptyset$.
- SO $cl\{a_i : i \in \mathbb{N}\} = \beta\mathbb{N}$.

TYCHONOFF, ČECH: $\beta\mathbb{N}$ CONTAINS NO NON-TRIVIAL CONVERGENT SEQUENCES.

LET $\langle x_n : n \in \mathbb{N}\rangle$ BE A SEQUENCE IN $\beta\mathbb{N}$ SUCH THAT $\{x_n : n \in \mathbb{N}\}$ IS RELATIVELY DISCRETE, FIND PAIRWISE DISJOINT A_n WITH $x_n \in \overline{A_n}$ FOR ALL n.
NEXT LET $E = \bigcup_n A_{2n}$ AND $O = \bigcup_n A_{2n-1}$
THEN $\overline{E} \cap \overline{O} = \emptyset$ HENCE $\overline{\{x_{2n} : n \in \mathbb{N}\}} \cap \overline{\{x_{2n-1} : n \in \mathbb{N}\}} = \emptyset$
SO THE SEQUENCE HAS (AT LEAST) TWO ACCUMULATION POINTS.

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CARDINALITY OF $\beta\mathbb{N}$.

COUNTABLE SET: $M = \{\langle n, s\rangle : n \in \mathbb{N},\ s \subseteq \mathcal{P}(\{1,2,\dots,n\})\}$

FOR $A \subseteq \mathbb{N}$ PUT $X_A = \{\langle n, s\rangle : X \cap \{1,2,\dots,n\} \in s\}$

GIVEN $A_1,\dots,A_k$, $B_1,\dots,B_l$ ALL DIFFERENT THE INTERSECTION

$$I = X_{A_1} \cap \dots \cap X_{A_k} \cap (M\setminus X_{B_1}) \cap \dots \cap (M \setminus X_{B_l})$$

IS INFINITE:

IF n IS SUCH THAT ALL $A_i \cap \{1,\dots,n\}$ AND $B_j \cap \{1,2,\dots,n\}$ ARE DISTINCT THEN

$$\langle n, \{A_i \cap \{1,2,\dots,n\} : i \leq k\}\rangle \in I$$

$\{X_A : A \subseteq \mathbb{N}\}$ IS AN INDEPENDENT FAMILY.

$$S : \beta M \longrightarrow \{0,1\}^{\mathcal{P}(\mathbb{N})} \qquad S(u)(A) = 1 \iff X_A \in u$$

DEFINES A (CONTINUOUS) SURJECTION

SO $|\beta\mathbb{N}| \geq 2^{|\mathcal{P}(\mathbb{N})|}$, BUT $\beta\mathbb{N} \subseteq \mathcal{P}(\mathcal{P}(\mathbb{N}))$ SO ALSO $|\beta\mathbb{N}| \leq 2^{|\mathcal{P}(\mathbb{N})|}$.

ACTUALLY: IF $F \subseteq \beta\mathbb{N}$ IS CLOSED THEN
EITHER F IS FINITE
OR $|F| = |\beta\mathbb{N}|$

$\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ IS A CLOSED SUBSPACE OF $\beta\mathbb{N}$
THE FAMILY $\{A^* : A \subseteq \mathbb{N}\}$ IS A BASE FOR $\mathbb{N}^*$,
WHERE $A^* = \overline{A}\setminus\mathbb{N}$

OBSERVE
$A^* \neq \emptyset \iff A$ IS INFINITE
$A^* = B^* \iff A \triangle B$ IS FINITE
$A^* \subseteq B^* \iff A \setminus B$ IS FINITE

IN FACT: $\{A^* : A \subseteq \mathbb{N}\}$ IS ISOMORPHIC TO $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$

MANY PROOFS IN $\mathbb{N}^*$ TAKE PLACE IN $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$
OR IN $\mathcal{P}(\mathbb{N})$ 'MOD FINITE'

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EXAMPLE: TAKE THE COUNTABLE SET $T = 2^{<\omega}$, I.E.,

T IS THE SET OF FINITE 0-1-SEQUENCES
T IS A TREE WHEN ORDERED BY EXTENSION:
IF $s, t \in T$ AND $s : m \to 2$ AND $t : n \to 2$
THEN $s \le t$ MEANS – $m \le n$
– $s(i) = t(i)$ FOR $i \in m$.

FOR $x : \omega \to 2$ LET $B_x = \{x \restriction m : m \in \omega\}$
NOTE $x \ne y \to B_x \cap B_y$ IS FINITE
LET i BE MINIMAL SUCH THAT $x(i) \ne y(i)$
THEN $\{x \restriction m : m > i\} \subseteq B_x \setminus B_y$
$\{y \restriction m : m > i\} \subseteq B_y \setminus B_x$
$\{x \restriction m : m \le i\} = \{y \restriction m : m \le i\} = B_x \cap B_y$

SO $B_x^* \cap B_y^* = \emptyset$ IF $x \ne y$
$B_x^* \ne \emptyset$ FOR ALL x

$\therefore \{B_x^* : x \in 2^\omega\}$ IS A FAMILY OF PAIRWISE DISJOINT CLOPEN SETS

LET $Q = \{x \in 2^\omega : (\exists n)(\forall i \ge n)(x(i) = x(n))\}$
$P = 2^\omega \setminus Q$

DEFINE $O_Q = \bigcup\{B_x^* : x \in Q\}$ AND $O_P = \bigcup\{B_x^* : x \in P\}$

THEN $\mathrm{cl}\, O_Q \cap \mathrm{cl}\, O_P \ne \emptyset$
. IF $\mathrm{cl}\, O_Q \cap \mathrm{cl}\, O_P = \emptyset$ THEN THERE IS $A \subseteq \mathbb{N}$ SUCH THAT $\mathrm{cl}\, O_P \subseteq A^*$ AND $\mathrm{cl}\, O_Q \cap A^* = \emptyset$
. TAKE $A \subseteq \mathbb{N}$ SUCH THAT $B_x^* \subseteq A^*$ FOR ALL $x \in P$.
FOR $x \in P$ TAKE n_x (MINIMAL) SUCH THAT
$\{x \restriction m : m \ge n_x\} \subseteq A$.
BAIRE CATEGORY THEOREM: THERE ARE $s \in T$ AND m SUCH THAT
$P_m = \{x \in P : n_x = m\}$
IS DENSE IN $\{x \in 2^\omega : s \le x\}$ (USE TOPOLOGY OF 2^ω).
TAKE $y \in Q$ SUCH THAT $s \le y$.
LET k BE ARBITRARY AND TAKE $x \in P_m$ SUCH THAT $x \restriction k = y \restriction k$
THEN $\{y \restriction i : m \le i \le k\} = \{x \restriction i : m \le i \le k\}$
$\subseteq A$
k WAS ARBITRARY SO $\{y \restriction i : m \le i\} \subseteq A$
AND SO $B_y \subseteq^* A$, OR $B_y^* \subseteq A^*$ AND $A^* \cap O_Q \ne \emptyset$.

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SO $\beta\mathbb{N}$ IS SEPARABLE BUT $\mathbb{N}^*$ IS NOT

$\beta\mathbb{N}$ IS EXTREMALLY DISCONNECTED BUT $\mathbb{N}^*$ IS NOT

$\mathbb{N}^*$ IS THE MORE INTERESTING OBJECT.

a) NONEMPTY G_δ-SETS HAVE NONEMPTY INTERIOR

b) DISJOINT OPEN F_σ-SETS HAVE DISJOINT CLOSURES

a: LET $G = \bigcap_n O_n$ BE A NONEMPTY G_δ-SET
TAKE $x \in G$ AND CHOOSE $A_n \subseteq \mathbb{N}$ SUCH THAT
- $x \in A_n^* \subseteq O_n$
- $A_{n+1}^* \subseteq A_n^*$

NOW CHOOSE NATURAL NUMBERS i_n SUCH THAT
$$i_n \in \bigcap_{m \le n} A_m \quad \text{AND} \quad i_{n+1} > i_n$$
FOR ALL n.
THEN $\{i_n : n \in \mathbb{N}\} \setminus A_m \subseteq \{i_n : n < m\}$
SO $\{i_n : n \in \mathbb{N}\}^* \subseteq A_m^*$ FOR ALL m SO
$\{i_n : n \in \mathbb{N}\}^* \subseteq G$.

b: IF $O = \bigcup_n F_n$ AND $U = \bigcup_n G_n$ ARE DISJOINT OPEN F_σ-SETS
THEN WE CAN FIND $\{A_n : n \in \mathbb{N}\}$ AND $\{B_n : n \in \mathbb{N}\}$
SUCH THAT
$$F_n \subseteq A_n^* \subseteq A_{n+1}^* \subseteq O$$
AND
$$G_n \subseteq B_n^* \subseteq B_{n+1}^* \subseteq U$$
FOR ALL n
NOTE $A_n^* \cap B_m^* = \emptyset$, HENCE $|A_n \cap B_m| < \omega$, FOR ALL n AND m
NOW WRITE
$$C_n = A_n \setminus \bigcup_{m \le n} B_m$$
$$D_n = B_n \setminus \bigcup_{m \le n} A_m$$
NOW $C_n^* = A_n^*$, $D_n^* = B_n^*$ BUT
$$C_n \cap D_m = \emptyset$$
FOR ALL n AND m
LET $C = \bigcup_n C_n$ AND $D = \bigcup_n D_n$
THEN $C \cap D = \emptyset$
$A_n^* \subseteq C^*$ ALL n
$B_n^* \subseteq D^*$ ALL n
$\therefore$ $O \subseteq C^*$ AND $U \subseteq D^*$ SO $\mathrm{cl}\, O \cap \mathrm{cl}\, U = \emptyset$.

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PAROVIČENKO'S THEOREM

1. IF X IS A COMPACT SPACE OF WEIGHT $\aleph_1$ (OR LESS) THEN X IS A CONTINUOUS IMAGE OF $\mathbb{N}^*$

OR (EQUIVALENTLY)

IF $\mathcal{B}$ IS A BOOLEAN ALGEBRA OF CARDINALITY $\aleph_1$ (OR LESS) THEN $\mathcal{B}$ CAN BE EMBEDDED INTO $\mathcal{P}(\omega)/\mathrm{FIN}$.

① EVERY COMPACT X IS A CONTINUOUS IMAGE OF A COMPACT ZERO-DIMENSIONAL SPACE Y OF THE SAME WEIGHT AS X.

PROOF LET $\mathcal{B}$ BE A BASE FOR THE TOPOLOGY OF X (OF CARDINALITY $w(X)$)

LET $\mathcal{B}^+$ BE THE BOOLEAN SUBALGEBRA OF $\mathcal{P}(X)$ GENERATED BY $\mathcal{B}$.

LET Y BE THE STONE SPACE OF X AND DEFINE

$\varphi : Y \to X$ BY

$\{\varphi(y)\} = \bigcap\{\mathrm{cl}\, A : A \in y\}$

- INTERSECTION IS NON EMPTY
- IF $x \in \bigcap\{\mathrm{cl} A : A \in y\}$ THEN $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\} \subseteq y$
- IF $x_1 \neq x_2$ THEN THERE ARE $B_1 \in \mathcal{B}_{x_1}$ AND $B_2 \in \mathcal{B}_{x_2}$ SUCH THAT $\mathrm{cl} B_1 \cap \mathrm{cl} B_2 = \emptyset$

  SO EITHER x_1 OR x_2 IS NOT IN $\bigcap\{\mathrm{cl} A : A \in y\}$
- CONTINUITY : EXERCISE

② ENUMERATE $\mathcal{B}$ AS $\{b_\alpha : \alpha < \omega_1\}$ WITH $b_0 = 0$ AND $b_1 = 1$.

LET $\mathcal{B}_\alpha$ BE THE SUBALGEBRA GENERATED BY $\{b_\beta : \beta < \alpha\}$

RECURSIVELY BUILD $h_\alpha : \mathcal{B}_\alpha \to \mathcal{P}(\omega)/\mathrm{FIN}$ EMBEDDING SUCH THAT h_α EXTENDS h_β IF $\beta < \alpha$.

- $\alpha = 0$, $\alpha = 1$ $\quad h_0(0) = 0 \quad h_0(1) = 1$

  $h_1 = h_0$
- α LIMIT $\quad h_\alpha = \bigcup_{\beta<\alpha} h_\beta$

- $\alpha \to \alpha+1$ IF $b_\alpha \in \mathcal{B}_\alpha$ LET $h_{\alpha+1} = h_\alpha$

  OTHERWISE DIVIDE $\mathcal{B}_\alpha$ INTO

  $L = \{x \in \mathcal{B}_\alpha : x < b_\alpha\}$

  $U = \{x \in \mathcal{B}_\alpha : b_\alpha < x\}$

  $M = \{x \in \mathcal{B}_\alpha : x \nleq b_\alpha \wedge b_\alpha \nleq x\} = \mathcal{B}_\alpha \setminus (L \cup U)$

  TO DEFINE $h_{\alpha+1}$ WE JUST NEED TO SPECIFY

  $c = h_{\alpha+1}(b_\alpha)$

  IT SHOULD SATISFY $\quad x \in L \to h_\alpha(x) < c$

  $x \in U \to c < h_\alpha(x)$

  $x \in M \to h_\alpha(x) \nleq c \wedge c \nleq h_\alpha(x)$

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THIS CAN BE DONE BUT IT IS QUITE TECHNICAL
IT USES a) AND b) ON PAGE 5.

(2) UNDER THE CONTINUUM HYPOTHESIS $\mathbb{N}^*$ IS CHARACTERIZED BY

- COMPACT, ZERO-DIMENSIONAL, NO ISOLATED POINTS
- NONEMPTY G_δ-SETS HAVE NONEMPTY INTERIORS
- DISJOINT OPEN F_σ-SETS HAVE DISJOINT CLOSURES

A LOOK AT $\beta\mathbb{R}$.

WE ONLY LOOK AT $\beta\mathbb{H}$, WHERE $\mathbb{H} = [0,\infty)$
WHY? $x \mapsto -x$ IS AN AUTOHOMEOMORPHISM OF $\mathbb{R}$
AND IT SHOWS THAT $\beta\mathbb{H}$ AND $\beta(-\infty,0]$
ARE THE SAME.

- $\beta\mathbb{H}$ IS CONNECTED : $\mathbb{H}$ IS CONNECTED AND DENSE IN $\beta\mathbb{H}$.
- SO, $\operatorname{cl}[n,\infty)$ IS ALSO CONNECTED FOR ALL n.
- HENCE $\mathbb{H}^* = \bigcap_{n=0}^{\infty} \operatorname{cl}[n,\infty)$ IS CONNECTED.

  THIS IS A GENERAL FACT: IF $\langle F_n : n\in\omega\rangle$ IS A DECREASING SEQUENCE OF COMPACT AND CONNECTED SETS THEN $\bigcap_{n\in\omega} F_n$ IS CONNECTED

- A NICE BASE FOR $\mathbb{H}^*$

LET F AND G BE CLOSED AND DISJOINT IN $\mathbb{H}^*$.
LET U AND V BE OPEN IN $\beta\mathbb{H}$ SUCH THAT
- $F \subseteq U$
- $G \subseteq V$
- $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$

WE BUILD SEQUENCES $\langle a_n : n\in\omega\rangle$, $\langle b_n : n\in\omega\rangle$, $\langle c_n : n\in\omega\rangle$ AND $\langle d_n : n\in\omega\rangle$ AS FOLLOWS
LET $a = \inf U$ AND $c = \inf V$, W.L.O.G. $a < c$.
$a_0 = \inf U$ $\quad c_0 = \inf V$
$a_1 = \inf U \cap (c_0,\infty)$ $\quad c_1 = \inf V \cap (a_1,\infty)$
$a_{n+1} = \inf U \cap (c_n,\infty)$ $\quad c_{n+1} = \inf V \cap (a_{n+1},\infty)$ ---
$b_n = \sup U \cap (a_n, c_n)$
$d_n = \sup V \cap (c_n, a_{n+1})$

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NOTE, BECAUSE $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$ WE ALWAYS HAVE
$\cdots\ a_n < b_n < c_n < d_n < a_{n+1} \cdots$

ALSO $\lim a_n = \infty$; FOR IF $\lim a_n = L < \infty$
THEN $L \in \mathrm{cl}\, U \cap \mathrm{cl}\, V$.

NEXT - $U \cap \mathbb{H} \subseteq \bigcup_{n=0}^{\infty} (a_n, b_n) = U^+$

- $V \cap \mathbb{H} \subseteq \bigcup_{n=0}^{\infty} (c_n, d_n) = V^+$

- SINCE $\bigcup_{n=0}^{\infty} [a_n, b_n] \cap \bigcup_{n=0}^{\infty} [c_n, d_n] = \emptyset$
WE ALSO HAVE
$\mathrm{cl}_\beta U^+ \cap \mathrm{cl}_\beta V^+ = \emptyset$

- SINCE $U \cap \mathbb{H} \subseteq U^+$ WE HAVE $U \subseteq \mathrm{Ex}\, U^+$
SINCE $V \cap \mathbb{H} \subseteq V^+$ WE HAVE $V \subseteq \mathrm{Ex}\, V^+$

- NOTE $\mathrm{Ex}\, O = \beta\mathbb{H} \setminus \mathrm{cl}_\beta(\mathbb{H} \setminus O)$ IS THE LARGEST OPEN SET IN $\beta\mathbb{H}$ WHOSE INTERSECTION WITH $\mathbb{H}$ IS O.

IN SHORT WE HAVE NICE OPEN SETS, U^+ AND V^+, SUCH THAT $F \subseteq \mathrm{Ex}\, U^+$, $G \subseteq \mathrm{Ex}\, V^+$
AND $\mathrm{cl}\,\mathrm{Ex}\, U^+ \cap \mathrm{cl}\,\mathrm{Ex}\, V^+ = \emptyset$.

ALSO: $F \subseteq \mathrm{cl}_\beta \bigcup_{n \in \omega} [a_n, b_n]$; $G \subseteq \mathrm{cl}_\beta \bigcup_{n \in \omega} [c_n, d_n]$

THE CLOSED SETS OF THE LATTER FORM CONSTITUTE A BASE FOR THE CLOSED SUBSETS OF $\mathbb{H}^*$

EXERCISE: USE THIS BASE TO PROVE THAT $\mathbb{H}^*$ HAS TWO PROPERTIES IN COMMON WITH $\mathbb{N}^*$:
- NONEMPTY G_δ-SETS HAVE NONEMPTY INTERIORS
- DISJOINT OPEN F_σ-SETS HAVE DISJOINT CLOSURES

ANOTHER PROPERTY:
EVERY COMPACT AND CONNECTED SPACE OF WEIGHT $\aleph_1$ OR LESS IS A CONTINUOUS IMAGE OF $\mathbb{H}^*$.

THE PROOF IS MUCH MORE INVOLVED THAN THAT OF PAROVIČENKO'S THEOREM: CONNECTEDNESS IS MUCH MORE COMPLICATED THAN ZERO-DIMENSIONALITY.

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CONTINUA IN $\mathbb{H}^*$.

LET K BE A PROPER SUBCONTINUUM OF $\mathbb{H}^*$;
SO: K IS COMPACT AND CONNECTED
$K \neq \mathbb{H}^*$
TAKE $x \in \mathbb{H}^* \setminus K$ AND FIND SEQUENCES AS ON PAGES ⑦ AND ⑧.
SO $x \in \mathrm{cl}\, \bigcup_n [c_n, d_n]$ AND $K \subseteq \bigcup_n [a_n, b_n]$.

NOW: CONSIDER $u = \{A \subseteq \omega : K \subseteq \mathrm{cl} \bigcup_{n\in A} [a_n, b_n]\}$
ABBREVIATE $F_A = \bigcup_{n \in A} [a_n, b_n]$

- $\emptyset \notin u$, $\omega \in u$
- IF $A \in u$ AND $A \subseteq B$ THEN $B \in u$
- IF $A, B \in u$ THEN $A \cap B \in u$
  NOTE $F_{A \setminus B} \cap F_{A \cap B} = \emptyset$ SO $\mathrm{cl}\, F_{A\setminus B} \cap \mathrm{cl}\, F_{A \cap B} = \emptyset$

  K IS CONNECTED SO $K \subseteq \mathrm{cl}\, F_{A \setminus B}$ OR $K \subseteq \mathrm{cl}\, F_{A \cap B}$
  BUT $\mathrm{cl}\, F_{A \setminus B} \cap \mathrm{cl}\, F_B = \emptyset$ SO $K \subseteq F_{A \cap B}$
- IF $A \subseteq \omega$ THEN $A \in u$ OR $\omega \setminus A \in u$.

SO u IS AN ULTRAFILTER AND

$$K \subseteq \bigcap_{A \in u} \mathrm{cl}\, F_A .$$

WE WRITE $F_u = \bigcap_{A \in u} \mathrm{cl}\, F_A$

- F_u IS COMPACT AND CONNECTED:
  SUPPOSE $F_u \subseteq U \cup V$ WITH U AND V OPEN AND DISJOINT.
  COMPACTNESS: THERE IS AN A SUCH THAT $\mathrm{cl}\, F_A \subseteq U \cup V$.
  EACH $[a_n, b_n]$ IS CONNECTED SO IF $n \in A$
  THEN $[a_n, b_n] \subseteq U$ OR $[a_n, b_n] \subseteq V$
  u IS AN ULTRAFILTER SO EITHER
  $B = \{n \in A : [a_n, b_n] \subseteq U\} \in u$ OR $C = \{n \in A : [a_n, b_n] \subseteq V\} \in u$
  THEREFORE $F_u \subseteq U$ OR $F_u \subseteq V$.

- F_u IS WHAT IS CALLED A STANDARD SUBCONTINUUM

- F_u HAS EMPTY INTERIOR IN $\mathbb{H}^*$.
  ASSUME WE HAVE A STANDARD OPEN SET $O = \bigcup_{n \in \omega} (c_n, d_n)$.
  TAKE A SEQUENCE $\langle n_i : i \in \omega \rangle$ (INCREASING)
  SO THAT FOR ALL i THERE IS m_i WITH
  $b_{n_i} < c_{m_i} < d_{m_i} < a_{n_{i+1}}$

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Let $A = \bigcup_{i\in\omega} [n_{2i}, n_{2i+1})$ and $B = \bigcup_{i\in\omega} [n_{2i+1}, n_{2i+2})$

If, say, $A \in u$ then $O_B = \bigcup_{i\in\omega} (c_{m_{2i+1}}, d_{m_{2i+1}})$
is disjoint from F_A
and so
$$\mathbb{H}^* \cap E \times O_B \subseteq \mathbb{H}^* \cap E \times O \setminus \operatorname{cl} F_A$$
which shows
$$\mathbb{H}^* \cap E \times O \nsubseteq F_u .$$

Corollary: $\mathbb{H}^*$ is an indecomposable continuum.

I.E. if K and L are proper subcontinua of $\mathbb{H}^*$
then $K \cup L \neq \mathbb{H}^*$.

We prove that $\mathbb{H}^*$ is hereditarily unicoherent,
i.e., if K and L are subcontinua then
$K \cap L$ is connected as well.

Let us consider two standard subcontinua:

F_u : determined by $\langle [a_n, b_n] : n \in \omega \rangle$ and $u \in \mathbb{N}^*$,

G_v : determined by $\langle [c_n, d_n] : n \in \omega \rangle$ and $v \in \mathbb{N}^*$.

We assume $F_u \cap G_v \neq \emptyset$

Let $A = \{ n : (\exists m)([a_n, b_n] \subseteq [c_m, d_m]) \}$

Assume $A \in u$ and define $f : A \to \omega$ in the
obvious way: $f(n) = m$ if $[a_n, b_n] \subseteq [c_m, d_m]$
Observe: if $B \in u$ then $f[B] \in v$
indeed $\emptyset \neq F_u \cap G_v \subseteq \operatorname{cl} \bigcup \{ [c_m, d_m] : m \in f[B] \} \cap G_v$
this implies $f[B] \cap C \neq \emptyset$ for all $C \in v$
and so $f[B] \in v$.
So $\{ f[B] : B \in u \} \subseteq v$
Also if $f[B] \cap C \neq \emptyset$ for all $B \in u$
then $f^{-1}[C] \in u$, for $B \cap f^{-1}[C] \neq \emptyset$ for $B \in u$.
This means that $\{ f[B] : B \in u \}$ generates v
and it implies that $F_u \subseteq G_v$

Similarly, if $\{ m : (\exists n)([c_m, d_m] \subseteq [a_n, b_n]) \} \in v$
then $G_v \subseteq F_u$.

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SO NOW WE HAVE THE CASE WHERE
$A = \{ n : (\forall m)([a_n, b_n] \nsubseteq [c_m, d_m]) \} \in u$
AND
$C = \{ m : (\forall n)([a_n, b_n] \nsupseteq [c_m, d_m]) \} \in v$

NOTE IF $B \in u$ AND $D \in v$ THEN
$$\emptyset \neq F_u \cap G_v \subseteq cl_\beta \bigcup_{n \in B} [a_n, b_n] \cap cl_\beta \bigcup_{m \in D} [c_m, d_m]$$
SO
$$\bigcup_{n \in B} [a_n, b_n] \cap \bigcup_{m \in D} [c_m, d_m] \neq \emptyset .$$

WE SHRINK THE SETS A AND C A BIT FURTHER.
FIRST WE ENSURE THAT FOR EVERY $n \in A$ THERE IS EXACTLY ONE $m \in C$ SUCH THAT
$[a_n, b_n] \cap [c_m, d_m] \neq \emptyset$
AND CONVERSELY FOR EVERY $m \in C$ THERE IS EXACTLY ONE $n \in A$ SUCH THAT $[c_m, d_m]$ MEETS $[a_n, b_n]$.
TO THIS END WE NOTE THAT EACH $[a_n, b_n]$ MEETS AT MOST TWO OF THE INTERVALS $[c_m, d_m]$: ONE AT EACH END AND THESE THEN CORRESPOND TO CONSECUTIVE MEMBERS OF C.
BY SYMMETRY THE SAME APPLIES TO THE INTERVALS INDEXED BY m.

WRITE $A = A_e \cup A_o$ AND $C = C_e \cup C_o$, WHERE IN EACH CASE X_e CONSISTS OF THE EVEN-NUMBERED MEMBERS OF X AND X_o OF ITS ODD-NUMBERED ONES.
WE REPLACE A BY A_e OR A_o, WHICHEVER BELONGS TO u AND C BY THE ONE OF C_e AND C_o THAT BELONGS TO v.

NEXT WE DIVIDE A INTO TWO SETS:
$A_l = \{ n : a_n \in \bigcup_{m \in C} [c_m, d_m] \}$
AND
$A_r = \{ n : b_n \in \bigcup_{m \in C} [c_m, d_m] \}$

WITHOUT LOSS OF GENERALITY $A_r \in u$.
NOW WE ARE IN THE SITUATION THAT FOR EVERY $n \in A_r$ THERE IS ONE $f(n) \in C$ SUCH THAT $a_n < c_{f(n)} \leq b_n \leq d_{f(n)}$
NOTE THAT
- $a_n < c_{f(n)}$ BECAUSE $[a_n, b_n] \nsubseteq [c_{f(n)}, d_{f(n)}]$
- a_n BELONGS TO NO OTHER INTERVAL $[c_m, d_m]$.

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THUS WE OBTAIN A MAP $f : A_2 \longrightarrow C$ SUCH THAT FOR ALL $X \in A_2$ WE HAVE

$$X \in u \iff f[X] \in v.$$

THIS THEN IMPLIES THAT

$$F_u \cup G_v = \bigcap_{X \in u} \mathrm{cl} \bigcup_{m \in X} [a_m, d_{f(m)}]$$

AND

$$F_u \cap G_v = \bigcap_{X \in u} \mathrm{cl} \bigcup_{m \in X} [c_{f(m)}, d_m]$$

THUS WE FIND THAT $F_u \cap G_v$ IS CONNECTED.

EXERCISE:

USE THIS METHOD TO SHOW: IF $\{F_i : i < n\}$ IS A FINITE FAMILY OF STANDARD SUBCONTINUA SUCH THAT $F_i \cap F_j \neq \emptyset$ FOR ALL $i, j < n$ THEN

$$\bigcap_{i<n} F_i$$

IS A NONEMPTY STANDARD SUBCONTINUUM.

NOW LET K AND L BE TWO SUBCONTINUA OF $\mathbb{H}^*$ THAT INTERSECT.
LET $\mathcal{F}_K$ BE THE FAMILY OF ALL STANDARD SUBCONTINUA THAT CONTAIN K AND LET $\mathcal{F}_L$ BE THE CORRESPONDING FAMILY FOR L.

BY WHAT WE SAW ON PAGE 9 WE HAVE

$$K = \bigcap \mathcal{F}_K \quad \text{AND} \quad L = \bigcap \mathcal{F}_L$$

SO $K \cap L = \bigcap(\mathcal{F}_K \cup \mathcal{F}_L)$

THE EXERCISE IMPLIES THAT THE INTERSECTION OF EVERY FINITE SUBFAMILY OF $\mathcal{F}_K \cup \mathcal{F}_L$ IS A STANDARD SUBCONTINUUM.

BUT THEN IT FOLLOWS THAT $K \cap L$ IS A CONTINUUM AS WELL.

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FURTHER READING

ALAN DOW AND K.P. HART

- THE ČECH-STONE COMPACTIFICATION
- THE ČECH-STONE COMPACTIFICATIONS OF $\mathbb{N}$ AND $\mathbb{R}$.

BOTH IN THE ENCYCLOPEDIA OF GENERAL TOPOLOGY.

K.P. HART

- THE ČECH-STONE COMPACTIFICATION OF THE REAL LINE
  IN RECENT PROGRESS IN GENERAL TOPOLOGY

ALL CAN BE FOUND AT

FA.ITS.TUDELFT.NL/~HART

J. VAN MILL

- AN INTRODUCTION TO $\beta\omega$
  IN HANDBOOK OF SET-THEORETIC TOPOLOGY.