Dimension(s) of compact F-spaces

Quidquid latine dictum sit, altum videtur

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Outline

Theorem

Let X be separable and metrizable and $n \in \mathbb{N}$. Then the dimension of X is at most n if and only if there are a zero-dimensional, separable and metrizable space Y and a closed continuous surjection $f : Y \to X$ such that $\left|f^{\leftarrow}(x)\right| \leqslant n + 1$ for all $x \in X$.

About the proofs

One direction uses the large inductive dimension.

Theorem

If Y is normal and strongly zero-dimensional and $f : Y \to X$ is closed, continuous and onto with $\left|f^{\leftarrow}(x)\right| \leqslant n+1$ for all $x \in X$ then $\operatorname{Ind} X \leqslant n$.

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About the proofs

Proof.

By induction (of course).
Given disjoint closed sets A and B in X find a closed set Z in Y such that $f[Z]$ is a partition between and $f \restriction Z$ has fibers of size at most n.
The speaker draws an instructive picture . . . $\square$

About the proofs

The other direction uses the covering dimension dimension.
$\dim X \leqslant n$ iff for every open cover $\mathcal{U}$ of X of cardinality $n+2$ there is an open refinement $\mathcal{V} = \{V_U : U \in \mathcal{U}\}$ with $\bigcap\{\operatorname{cl} V : V \in \mathcal{V}\} = \emptyset$.
Refinement: $V_U \subseteq U$ for all U and $\bigcup \mathcal{V} = X$.

About the proofs

Theorem

If X is compact and metrizable with $\dim X \leqslant n$ then there are a zero-dimensional, compact and metrizable space Y and a continuous surjection $F : Y \to X$ with $\left|f^{\leftarrow}(x)\right| \leqslant n+1$ for all $x \in X$.

Proof.

Idea of proof: make finite closed covers of order at most $n+1$; give these the discrete topology; take their product and let Y be a suitable subspace of that product. $\square$

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What makes this work?

The reason we have an equivalence is the fundamental fact from dimension theory that $\dim X = \operatorname{ind} X = \operatorname{Ind} X$ for all separable and metrizable X.
And the compactification theorem: a separable and metrizable space has a metric compactification with the same dimension(s).

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F-spaces of weight $\mathfrak{c}$

Experience has taught us that compact F-spaces of weight $\mathfrak{c}$ behave in many ways like compact metrizable spaces, *provided the Continuum Hypothesis holds*

Remember: X is an F-space if every finitely generated ideal in $C^*(X)$ is principal.

Or, somewhat more topological: disjoint cozero sets are completely separated.

Or, for normal spaces: disjoint cozero sets have disjoint closures.

Equality of dimensions

Theorem (CH)

For every compact F-space, X, of weight $\mathfrak{c}$ we have

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X$$

Proof

The inequalities $\dim X \leqslant \operatorname{ind} X \leqslant \operatorname{Ind} X$ hold for *every* compact space.

Proof, continued

Proof

The interesting part is the proof of $\operatorname{Ind} X \leqslant \dim X$.
Given disjoint closed sets A and B we build a partition, L, between them with $\dim L \leqslant \dim X - 1$.
How: we have $\aleph_1$ many potential basic open covers of L of size $\dim X + 1$; enumerate them: $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$.
Build increasing sequences $\langle C_\alpha : \alpha < \omega_1 \rangle$ and $\langle D_\alpha : \alpha < \omega_1 \rangle$ of cozero sets, with $C_\alpha \cap D_\alpha = \emptyset$ for all α.

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Proof, continued

Proof

At stage α, check if $C_\alpha \cup D_\alpha \cup \bigcup \mathcal{U}_\alpha = X$.
In that case take a refinement $\{O\} \cup \mathcal{V}_\alpha$ of $\{C_\alpha \cup D_\alpha\} \cup \mathcal{U}_\alpha$ whose closures have empty intersection.
Take $C_{\alpha+1}$ and $D_{\alpha+1}$ such that $C_\alpha \cup \bigcap_{U \in \mathcal{U}_\alpha} \operatorname{cl} V_U \subseteq C_{\alpha+1}$ and $D_\alpha \subseteq D_{\alpha+1}$.
Apart from some technicalities this works.

A very general theorem

Theorem (CH)

Let X be a compact F-space of weight $\mathfrak{c}$. Then X has a base $\{B_\alpha : \alpha < \omega_1\}$ with the following property: whenever F is a finite subset of ω_1 the intersection

$$\bigcap_{\alpha \in F} \operatorname{Fr} B_\alpha$$

has dimension at most $\dim \operatorname{Fr} B_{\min F} - |F| + 1$.

About the proof

It uses a simultaneous version of the proof of $\operatorname{Ind} X \leqslant \dim X$.

In the separable metric case one can build a partition, L, such that

$$\dim(L \cap D) \leqslant \dim D - 1$$

for countably many closed sets D at once.

In the case of a compact F-space of weight $\mathfrak{c}$, assuming CH, you can do this in one go for $\aleph_1$ many closed sets.

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A special case

Theorem (CH)

Let X be a compact F-space of weight $\mathfrak{c}$ and dimension n. Then X has a base $\{B_\alpha : \alpha < \omega_1\}$ with the following property:

$$\bigcap_{\alpha \in F} \operatorname{Fr} B_\alpha = \emptyset$$

whenever F is a subset of ω_1 with $n + 1$ elements.

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A finite-to-one map

We may assume our base consists of regular open sets ($B_\alpha = \operatorname{int} \operatorname{cl} B_\alpha$).
Take the Boolean subalgebra, B, of $\mathrm{RO}(X)$ generated by our base.
Then the natural map from the Stone space of B onto X is (at most) 2^n-to-one.

A finite-to-one map

Bummer! $2^n > n + 1$ (when $n \geqslant 2$).

We have an other proof, with the same result: 2^n.

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An example

The first question that should occur to everyone has an answer:

There is a compact F-space of weight $\mathfrak{c}^+$ with non-coinciding dimensions (my student Jan van Mill).

This parallels the 'classic' case: there are compact spaces of weight $\aleph_1$ with non-coinciding dimensions.

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What if CH fails?

The second question that should occur to everyone has no answer (yet).
One possibility: there are many compact spaces of weight $\mathfrak{c}$ with non-coinciding dimensions.

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What if CH fails?

Take such a space, X, for example with $\dim X = 1$ and $\operatorname{ind} X = \operatorname{Ind} X = 2$.
Consider $Y = \omega \times X$ and $Y^* = \beta Y \setminus Y$.
By our first result we have $\dim Y^* = \operatorname{ind} Y^* = \operatorname{Ind} Y^*$ if CH holds.

What if CH fails?

Last year's tutorial: $\dim C = \dim X = 1$ for every component C of Y^*.

Also $\dim Y^* \leqslant \dim \beta Y = 1$, so $\dim Y^* = 1$.

Hence(!): $\operatorname{Ind} Y^* = 1 < 2 = \operatorname{Ind} \beta Y$ (if CH).

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What can be said if CH fails? In particular models where CH fails.

Could it be that $\operatorname{Ind} Y^* = 2$ in some such model?

There are many X to play with.

What with 2^n?

The third question on everyone's lips:
can 2^n be brought down to $n+1$?

(As it should be.)
We have no idea.

Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart, J. van Mill,
Covering dimension and finite-to-one maps, Topology and its Applications, **158** (2011), 2512–2519.

J. van Mill,
A compact F-space with noncoinciding dimensions, Topology and its Applications **159** (2012), 1625–1633.

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