A chain condition for operators from $C(K)$-spaces

Quidquid latine dictum sit, altum videtur

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Outline

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Pełczyński's Theorem

Confusingly (for a topologist):

- K generally denotes a compact space,
- X generally denotes a Banach space.

Theorem

An operator $T : C(K) \to X$ is weakly compact iff there is no isomorphic copy of c_0 on which T is invertible.

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Reformulation

An operator $T : C(K) \to X$ is *not* weakly compact iff there is a sequence $\langle f_n : n < \omega \rangle$ of continuous functions such that

- $\|f_n\| \leqslant 1$ for all n
- $\operatorname{supp} f_m \cap \operatorname{supp} f_n = \emptyset$ whenever $m \neq n$
- $\inf_n \|Tf_n\| > 0$

Where's the chain?

First: an order on $C(K)$.
We say $f \prec g$ if

- $f \neq g$
- $g \restriction \operatorname{supp} f = f \restriction \operatorname{supp} f$

Second: another order on $C(K)$.
Let $\delta > 0$; we say $f \prec_\delta g$ if

- $\|g - f\| \geqslant \delta$
- $g \restriction \operatorname{supp} f = f \restriction \operatorname{supp} f$

The speaker draws an instructive picture.

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Here's the chain

An operator $T : C(K) \to X$ is *not* weakly compact iff there is an infinite $\prec$-chain, C, such that

$$\inf\left\{\|Tf - Tg\| : \{f, g\} \in [C]^2\right\} > 0$$

Proof.

Given $\langle f_n : n < \omega\rangle$ let $g_n = \sum_{i \leqslant n} f_i$; then $\langle g_n : n < \omega\rangle$ is a (bad) chain.
Given an infinite chain, C, take a monotone sequence $\langle g_n : n < \omega\rangle$ in C and let $f_n = g_{n+1} - g_n$ for all n. $\square$

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Here is the chain condition

For every uncountable $\prec$-chain in $C(K)$ we have

$$\inf\left\{\|f-g\| : \{f,g\} \in [C]^2\right\} = 0$$

In other words:

For every $\delta > 0$: every $\prec_\delta$-chain is countable.

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Why 'uncountable'?

Well, . . .

Theorem

If K is extremally disconnected then $T : C(K) \to X$ is weakly compact iff

$$\inf\{\|Tf - Tg\| : \{f, g\} \in [C]^2\} = 0$$

for every uncountable $\prec$-chain C.

In fact if T is not weakly compact then we can find a $\prec$-chain isomorphic to $\mathbb{R}$ where the infimum is positive, that is, there are a $\delta > 0$ and a $\prec_\delta$-chain isomorphic to $\mathbb{R}$.

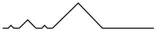
$\prec$-chains are easy

Uncountable $\prec$-chains are quite ubiquitous:

Example

There is an uncountable $\prec$-chain in $C\big([0,1]\big)$.
Start with $f : x \mapsto d(x,\mathbb{C})$, where $\mathbb{C}$ is the Cantor set.
For $t \in \mathbb{C}$ let $f_t = f \cdot \chi_{[0,t]}$, then $\{f_t : t \in \mathbb{C}\}$ is a $\prec$-chain.

Do we need an instructive picture?

$f_{\frac{2}{3}}$ 

♗ is not an antichain condition

The *separable*(!) double-arrow space $\mathbb{A}$ has a $\prec_1$-chain that is isomorphic to $\mathbb{R}$.

Remember: we have $\mathbb{A} = \bigl((0,1] \times \{0\}\bigr) \cup \bigl([0,1) \times \{1\}\bigr)$ ordered lexicographically.
For $t \in (0,1)$ let f_t be the characteristic function of the interval $\bigl[\langle 0,1\rangle, \langle t,0\rangle\bigr]$.

Time for another instructive picture.

A few observations

Let C be a $\prec$-chain; for $f \in C$ put

$$S(f, C) = \{x : f(x) \neq 0\} \setminus \bigcup\{\operatorname{supp} g : g \in C, g \prec f\}$$

Note: in the example in $C\bigl([0,1]\bigr)$ there are f_t , e.g. $f_{\frac{1}{3}}$, with $S(f_t) = \emptyset$, whereas $S(f_{\frac{2}{3}}) = (\frac{1}{3}, \frac{2}{3})$.
In the chain in $C(\mathbb{A})$ we have $S(f_t) = \bigl\{\langle t, 0\rangle\bigr\}$ for all t.

A useful lemma

From now on all functions are positive.

Lemma

If C is a $\prec_\delta$-chain for some $\delta > 0$ then $S(f, C) \neq \emptyset$ for all $f \in C$; in fact there is $x \in S(f, C)$ with $f(x) \geqslant \delta$.

Proof.

Clear if f has a direct predecessor.
Otherwise let $\langle g_\alpha : \alpha < \theta \rangle$ be increasing and cofinal in $\{g \in C : g \prec f\}$.
Pick $x_\alpha \in \operatorname{supp} g_{\alpha+1} \setminus \operatorname{supp} g_\alpha$ with $g_{\alpha+1}(x) \geqslant \delta$.
Any cluster point, x, of $\langle g_\alpha : \alpha < \theta \rangle$ will satisfy $f(x) \geqslant \delta$ and $g(x) = 0$ for all $g \prec f$. □

The convergent sequence

$C(\omega+1)$ has an uncountable $\prec$-chain.
Let $b:\omega\to\mathbb{Q}$ be a bijection. For $t\in\mathbb{R}$ define f_t by

$$f_t(\alpha)=\begin{cases}2^{-\alpha} & \text{if } b(\alpha)<t\\ 0 & \text{otherwise.}\end{cases}$$

If $\delta>0$ then every $\prec_\delta$-chain in $C(\omega+1)$ is countable.

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Another lemma

Lemma

If K is locally connected and if C is a $\prec_\delta$-chain for some $\delta > 0$ then $S(f, C)$ is (nonempty and) open.

Proof.

Let $x \in S(f, C)$ and let U be a connected neighbourhood of x such that $f(y) > \frac{1}{2}f(x)$ for all $y \in U$. We claim $U \cap \operatorname{supp} g = \emptyset$ if $g \prec f$.

Indeed if $U \cap \operatorname{supp} g \neq \emptyset$ then U meets the boundary of $\operatorname{supp} g$ and then we find $y \in U$ such that $f(y) = g(y) = 0$. $\square$

More small $\prec_\delta$-chains

If K is locally connected then every $\prec_\delta$-chain has cardinality at most $c(K)$ (cellularity of K).

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A closer look at local connectivity

We assume K is locally connected (and that $\delta > 0$).

Lemma

There is no increasing $\prec_\delta$-chain of order type $\omega + 1$.

Proof.

Let $\langle f_n : n < \omega \rangle$ be increasing with respect to $\prec_\delta$ and assume f is a $\prec_\delta$ upper bound.
For each n let $A_n = \{y : f_{n+1}(y) \geqslant \delta, f_n(y) = 0\}$ and let x be a cluster point of $\{A_n : n < \omega\}$.
Because $f(y) = f_{n+1}(y) \geqslant \delta$ if $y \in A_n$ we find $f(x) \geqslant \delta$. $\square$

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A closer look at local connectivity

We assume K is locally connected (and that $\delta > 0$).

Lemma

There is no increasing $\prec_\delta$-chain of order type $\omega + 1$.

Proof: continued.

Let U be a neighbourhood of x such that $f(y) > \frac{1}{2}\delta$ for all $y \in U$. This shows U has many clopen pieces: $B_n \cap U$, whenever $A_n \cap U \neq \emptyset$; here $B_n = \{y : f_{n+1}(y) > 0, f_n(y) = 0\}$. □

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A closer look at local connectivity

We still assume K is locally connected (and that $\delta > 0$).

Lemma

There is no decreasing $\prec_\delta$-chain of order type $\omega^\star$.

More or less the same proof, with

$$A_n = \{y : f_n(y) \geqslant \delta, f_{n+1}(y) = 0\}$$

and

$$B_n = \{y : f_n(y) > 0, f_{n+1}(y) = 0\}$$

A structural result

If K is locally connected then $\prec_\delta$ is a well-founded relation.
All chains have order type (at most) ω.

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Further examples

One-point compactifications of discrete spaces have property ♗.

One-point compactifications of ladder system spaces have property ♗.

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My favourite continuum

$\mathbb{H} = [0, \infty)$ and $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$.
$\mathbb{H}^*$ is a continuum that is indecomposable and hereditarily unicoherent.
$C(\mathbb{H}^*)$ does not have property ♗.

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How to make an uncountable $\prec_\delta$-chain

Start with a sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ in $\prod_{n\in\omega} 2^n$ with the property that $\lim_{n\to\omega} h_\beta(n) - h_\alpha(n) = \infty$.

Then make a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ of continuous functions on $\mathbb{M} = \omega \times [0,1]$ such that:

If $\beta < \alpha$ then there is an N such that for all $n \geqslant N$ the function $f_\alpha(n,x)$

- increases from 0 to 1 on $[h_\beta(n)2^{-n}, (h_\beta(n)+1)2^{-n}]$
- is constant 1 on $[(h_\beta(n)+1)2^{-n}, (h_\beta(n)+2)2^{-n}]$
- decreases from 1 to 0 on $[(h_\beta(n)+2)2^{-n}, (h_\beta(n)+3)2^{-n}]$

Everywhere else f_α will be zero.

How to make an uncountable $\prec_\delta$-chain

For every α we let $f^*\alpha = \beta f_\alpha \restriction \mathbb{M}^*$.
Then $\langle f^*_\alpha : \alpha < \omega_1 \rangle$ is a $\prec_1$-chain in $C(\mathbb{M}^*)$.

$\mathbb{H}^*$ is a simple quotient of $\mathbb{M}^*$ and the chain is transferred painlessly to $C(\mathbb{H}^*)$.

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Question

What (classes of) spaces have property ♗?

Light reading

Website: `fa.its.tudelft.nl/~hart`

Klaas Pieter Hart, Tomasz Kania and Tomasz Kochanek.
A chain condition for operators from $C(K)$-spaces, The Quarterly Journal of Mathematics (2013), DOI:10.1093/qmath/hat006

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