A chain condition for operators from $C(K)$-spaces

Quidquid latine dictum sit, altum videtur

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Outline

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1. Weakly compact operators
2. A chain condition
3. Spaces with and without uncountable $\prec_\delta$-chains
4. Sources

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Pełczyński's Theorem

Confusingly (for a topologist):

- K generally denotes a compact space,
- X generally denotes a Banach space.

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Theorem

An operator $T : C(K) \to X$ is weakly compact iff there is no isomorphic copy of c_0 on which T is invertible.

Reformulation

An operator $T : C(K) \to X$ is *not* weakly compact iff there is a sequence $\langle f_n : n < \omega \rangle$ of continuous functions such that

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- $\inf_n \|Tf_n\| > 0$

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Proof.

Given $\langle f_n : n < \omega\rangle$ let $g_n = \sum_{i \leqslant n} f_i$; then $\langle g_n : n < \omega\rangle$ is a (bad) chain.

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Given $\langle f_n : n < \omega\rangle$ let $g_n = \sum_{i\leqslant n} f_i$; then $\langle g_n : n < \omega\rangle$ is a (bad) chain.

Given an infinite chain, C, take a monotone sequence $\langle g_n : n < \omega\rangle$ in C and let $f_n = g_{n+1} - g_n$ for all n. □

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In other words:

For every $\delta > 0$: every $\prec_\delta$-chain is countable.

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Theorem

If K is extremally disconnected then $T : C(K) \to X$ is weakly compact iff

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In fact if T is not weakly compact then we can find a $\prec$-chain isomorphic to $\mathbb{R}$ where the infimum is positive, that is, there are a $\delta > 0$ and a $\prec_\delta$-chain isomorphic to $\mathbb{R}$.

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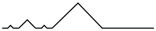
$f_{\frac{2}{3}}$ 

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The *separable*(!) double-arrow space $\mathbb{A}$ has a $\prec_1$-chain that is isomorphic to $\mathbb{R}$.

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Time for another instructive picture.

A few observations

Let C be a $\prec$-chain; for $f \in C$ put

$$S(f, C) = \{x : f(x) \neq 0\} \setminus \bigcup \{\operatorname{supp} g : g \in C, g \prec f\}$$

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In the chain in $C(\mathbb{A})$ we have $S(f_t) = \bigl\{\langle t, 0\rangle\bigr\}$ for all t.

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Pick $x_\alpha \in \operatorname{supp} g_{\alpha+1} \setminus \operatorname{supp} g_\alpha$ with $g_{\alpha+1}(x) \geqslant \delta$.

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Any cluster point, x, of $\langle g_\alpha : \alpha < \theta \rangle$ will satisfy $f(x) \geqslant \delta$ and $g(x) = 0$ for all $g \prec f$. □

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Let $b : \omega \to \mathbb{Q}$ be a bijection. For $t \in \mathbb{R}$ define f_t by

$$f_t(\alpha) = \begin{cases} 2^{-\alpha} & \text{if } b(\alpha) < t \\ 0 & \text{otherwise.} \end{cases}$$

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If $\delta>0$ then every $\prec_\delta$-chain in $C(\omega+1)$ is countable.

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Let $x \in S(f, C)$ and let U be a connected neighbourhood of x such that $f(y) > \frac{1}{2}f(x)$ for all $y \in U$. We claim $U \cap \operatorname{supp} g = \emptyset$ if $g \prec f$.

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Indeed if $U \cap \operatorname{supp} g \neq \emptyset$ then U meets the boundary of $\operatorname{supp} g$ and then we find $y \in U$ such that $f(y) = g(y) = 0$. $\square$

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More small $\prec_\delta$-chains

If K is locally connected then every $\prec_\delta$-chain has cardinality at most $c(K)$ (cellularity of K).

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A closer look at local connectivity

We assume K is locally connected (and that $\delta > 0$).

Lemma

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For each n let $A_n = \{y : f_{n+1}(y) \geqslant \delta, f_n(y) = 0\}$ and let x be a cluster point of $\{A_n : n < \omega\}$.

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Because $f(y) = f_{n+1}(y) \geqslant \delta$ if $y \in A_n$ we find $f(x) \geqslant \delta$. □

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Let U be a neighbourhood of x such that $f(y) > \frac{1}{2}\delta$ for all $y \in U$.

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Let U be a neighbourhood of x such that $f(y) > \frac{1}{2}\delta$ for all $y \in U$. This shows U has many clopen pieces: $B_n \cap U$, whenever $A_n \cap U \neq \emptyset$; here $B_n = \{y : f_{n+1}(y) > 0, f_n(y) = 0\}$. □

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and

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A structural result

If K is locally connected then $\prec_\delta$ is a well-founded relation.

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If K is locally connected then $\prec_\delta$ is a well-founded relation.
All chains have order type (at most) ω.

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Further examples

One-point compactifications of discrete spaces have property ♗.

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One-point compactifications of ladder system spaces have property ♗.

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$C(\mathbb{H}^*)$ does not have property ♗.

How to make an uncountable $\prec_\delta$-chain

Start with a sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ in $\prod_{n\in\omega} 2^n$ with the property that $\lim_{n\to\omega} h_\beta(n) - h_\alpha(n) = \infty$.

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- increases from 0 to 1 on $[h_\beta(n)2^{-n}, (h_\beta(n)+1)2^{-n}]$
- is constant 1 on $[(h_\beta(n)+1)2^{-n}, (h_\beta(n)+2)2^{-n}]$
- decreases from 1 to 0 on $[(h_\beta(n)+2)2^{-n}, (h_\beta(n)+3)2^{-n}]$

Everywhere else f_α will be zero.

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For every α we let $f^*\alpha = \beta f_\alpha \restriction \mathbb{M}^*$.

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$\mathbb{H}^*$ is a simple quotient of $\mathbb{M}^*$ and the chain is transferred painlessly to $C(\mathbb{H}^*)$.

Question

What (classes of) spaces have property ♗?

Outline

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Light reading

Website: `fa.its.tudelft.nl/~hart`

Klaas Pieter Hart, Tomasz Kania and Tomasz Kochanek.
A chain condition for operators from $C(K)$-spaces, The Quarterly Journal of Mathematics (2013), DOI:10.1093/qmath/hat006

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