Subcontinua of $\mathbb{H}^*$

Non impeditus ab ulla sciencia

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The competition

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Outline

Main result

Theorem (Alan Dow, Y. T.)

The Čech-Stone remainder of the half line $\mathbb{H}$ has a family of $2^{\mathfrak{c}}$ many mutually nonhomeomorphic subcontinua.

The rest of this talk will consist of an explanation of all the terms and of a sketch of the proof.

The

Definite article, used to refer to a unique person or thing

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Čech-Stone

Čech-Stone compactification

Every completely regular space, X, has a compactification, βX, with the following characterizing property:
Every *bounded* continuous function $f : X \to \mathbb{R}$ has a continuous extension $\beta f : \beta X \to \mathbb{R}$.

Compactification: a compact Hausdorff space that contains (a homeomorphic copy of) X as a dense subspace.

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Čech-Stone

Čech-Stone remainder

We call $\beta X \setminus X$ the Čech-Stone remainder of X and denote it X^*.

Some people say ‘growth’, that sounds more logical but . . .
since when do we give ideas logical names?

The half line $\mathbb{H}$

The half line $\mathbb{H}$ is the interval $[0, \infty)$ in $\mathbb{R}$.
We study $\mathbb{H}^*$ because $\mathbb{R}^*$ is just the sum of two copies of $\mathbb{H}^*$.

All you need to know about $\beta\mathbb{H}$

As $\mathbb{H}$ is normal we have the following (again) characterizing property of $\beta\mathbb{H}$:
if F and G are closed and disjoint in $\mathbb{H}$
then their closures in $\beta\mathbb{H}$ are disjoint.

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Crucial property

Very nice open sets

Let F and G be closed and disjoint in $\mathbb{H}^*$. There are two sequences of open intervals, $\langle (a_n, b_n) : n \in \omega \rangle$ and $\langle (c_n, d_n) : n \in \omega \rangle$, such that

- $b_n < c_n$ and $d_n < a_{n+1}$ for all n (or the other way round);
- $F \subseteq \operatorname{Ex} U$ and $G \subseteq \operatorname{Ex} V$, where $U = \bigcup_n (a_n, b_n)$ and $V = \bigcup_n (c_n, d_n)$.

Notation: $\operatorname{Ex} U = \beta\mathbb{H} \setminus \mathrm{cl}_\beta(\mathbb{H} \setminus U)$ (the largest open subset of $\beta\mathbb{H}$ that extends U).

Indication of proof

Take open sets O_F and O_G, in $\beta\mathbb{H}$, around F and G respectively with disjoint closures ($\beta\mathbb{H}$ is normal).

Assume $\inf O_F < \inf O_G$.

- Let $a_0 = \inf O_F$ and $b_0 = \sup\{x \in O_F : (a_0, x) \cap O_G = \emptyset\}$.
- Let $c_0 = \inf\{x \in O_G : b_0 < x\}$ and $d_0 = \sup\{x \in O_G : (c_0, x) \cap O_F = \emptyset\}$.
- Let $a_1 = \inf\{x \in O_F : d_0 < x\}$ and $b_1 = \sup\{x \in O_F : (a_1, x) \cap O_G = \emptyset\}$.

Keep alternating

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About U and V

Note: $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$, hence $\operatorname{cl}_\beta \operatorname{Ex} U \cap \operatorname{cl}_\beta \operatorname{Ex} V = \emptyset$.

The sets of the form $\mathbb{H}^* \cap \operatorname{Ex} U$ form a base for the topology of $\mathbb{H}^*$.

And dually: the sets of the form $\mathbb{H}^* \cap \operatorname{cl}_\beta \bigcup_n [a_n, b_n]$ form a base for the closed sets of $\mathbb{H}^*$.

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a family

a: indefinite article, a broken down form of ‘an’
family: synonym for ‘set’

‘a family’ because we do not know whether we have them all

$2^{\mathfrak{c}}$

Traditionally: $\mathfrak{c} = |\mathbb{R}|$ and so $2^{\mathfrak{c}}$ is the cardinality of the power set of $\mathbb{R}$.

Well known: $\mathbb{R}$ has $\mathfrak{c}$ many closed sets, hence $\mathbb{H}^*$ has at most $2^{\mathfrak{c}}$ many closed sets (each closed set, K, is determined by $\{F : K \subseteq\in \operatorname{cl}_\beta F\}$).

Also well known: $\mathbb{H}^*$ contains ω^* and ω^* has $2^{\mathfrak{c}}$ many points, so $\mathbb{H}^*$ has exactly $2^{\mathfrak{c}}$ many points.

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Continuum

Easy: a continuum is a compact and connected Hausdorff space.

$\mathbb{H}^*$ is a continuum:

- $\mathbb{H}$ is connected, hence so is $\beta\mathbb{H}$;
- $\mathrm{cl}_\beta[n,\infty)$ is connected (for all n)
- $\mathbb{H}^* = \bigcap \mathrm{cl}_\beta[n,\infty)$ is connected

Exercise: a decreasing sequence of compact connected sets has a compact and connected intersection.

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Subcontinua of $\mathbb{H}^*$

Take a sequence $\langle [a_n, b_n] : n \in \omega \rangle$ of closed intervals (with $b_n < a_{n+1}$ and $\lim_n a_n = \infty$).
Take $x \in \mathbb{H}^*$.
Let u be the family of subsets, A, of ω that satisfy

$$x \in \operatorname{cl}_\beta \bigcup_{n \in A} [a_n, b_n]$$

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Subcontinua of $\mathbb{H}^*$

u is an ultrafilter

The intersection

$$[a_u, b_u] = \bigcap_{A\in u} \mathrm{cl}_\beta \bigcup_{n\in A} [a_n, b_n]$$

is (compact and) connected

It's what we call a standard subcontinuum of $\mathbb{H}^*$.

In fact . . .

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Subcontinua of $\mathbb{H}^*$

... every subcontinuum is the intersection of standard subcontinua.

Repeat previous arguments: if $x \notin K$ find a nice closed set, $F = \bigcup_n [a_n, b_n]$, such that $K \subseteq \mathrm{cl}_\beta F$ and $x \notin \mathrm{cl}_\beta F$.

As above, $u = \{A : K \subseteq \mathrm{cl}_\beta \bigcup_{n \in A} [a_n, b_n]\}$ is an ultrafilter, $K \subseteq [a_u, b_u]$, and $x \notin [a_u, b_u]$

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Indecomposability

Not too difficult: $[a_u, b_u]$ has empty interior in $\mathbb{H}^*$.

Corollary: if K and L are two *proper* subcontinua of $\mathbb{H}^*$ then $K \cup L \neq \mathbb{H}^*$.

In other words: $\mathbb{H}^*$ is an indecomposable continuum. (Bellamy, Woods).

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Standard subcontinua of $\mathbb{H}^*$

We need a model: every nice closed set looks like

$$\mathbb{M} = \omega \times [0,1]$$

- its closure looks like $\beta\mathbb{M}$
- the projection $\pi : \mathbb{M} \to \omega$ extends to $\beta\pi : \beta\mathbb{M} \to \beta\omega$
- for $u \in \omega^*$ the set $[a_u, b_u]$ looks like $\beta\pi^{\leftarrow}(u)$
- we write $\mathbb{I}_u$ for this preimage.

Properties

The continuum $\mathbb{I}_u$

- is irreducible between 0_u and 1_u
- contains the ultrapower $(0,1)^\omega/u$ as a subspace (with its order topology)
- is itself not linearly ordered

The points of $(0,1)^\omega/u$ are cut points of $\mathbb{I}_u$ but ...

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Properties

... if $\langle x_n : n \in \omega \rangle$ is an increasing sequence of such cut points then its 'supremum' is a non-trivial continuum.

More generally: if I is an initial segment of $(0,1)^\omega/u$ then $\sup I$ is either a cut point or an indecomposable continuum (so certainly the latter if I has countable cofinality).

We call such continua *layers* of $\mathbb{I}_u$.
These layers will be important later on.

Further properties

Let $[c_v, d_v]$ and $[a_u, b_u]$ be standard subcontinua (given by sequences $\bigl\langle [c_n, d_n] : n \in \omega \bigr\rangle$ and $\bigl\langle [a_n, b_n] : n \in \omega \bigr\rangle$ respectively).

Then $[c_v, d_v] \subseteq [a_u, b_u]$ iff the (partial) function

$$\varphi = \bigl\{ \langle m, n \rangle : [c_m, d_m] \subseteq [a_n, b_n] \bigr\}$$

satisfies $\varphi(v) = u$ (so, implicitly, $\operatorname{dom} \varphi \in v$ and $\operatorname{ran} \varphi \in u$).

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Further properties

Two cases:

- φ is one-to-one on some member of v, then $[c_v, d_v]$ is a subinterval of $[a_u, b_u]$
- φ is one-to-one on no member of v, then $[c_v, d_v]$ is a subset of some layer of $[a_u, b_u]$

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Further properties

A technical result.

Lemma

Let K and L be two subcontinua of $\mathbb{H}^*$ that intersect and such that (at least) one of them is indecomposable.
Then $K \subseteq L$ or $L \subseteq K$.

For the proof see the references at the end.

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CH fails

Theorem (Alan Dow, $\neg$CH)

There exists a family of $2^{\mathfrak{c}}$ mutually non-homeomorphic standard subcontinua.

Proof.

Based on a result of Shelah's on the existence of a family of $2^{\mathfrak{c}}$ mutually non-isomorphic ultrapowers of $(0, 1)$. $\square$

CH holds

CH implies that all standard subcontinua are homeomorphic, so there goes that idea.

We find $2^{\mathfrak{c}}$ mutually non-homeomorphic indecomposable subcontinua.

A byproduct of our construction is a family of $2^{\mathfrak{c}}$ mutually non-homeomorphic decomposable subcontinua.

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Main ingredient

Γ is the set of all sequences $\langle [a_n, b_n] : n \in \omega \rangle$ of closed intervals, with $a_n, b_n \in \omega$ and $a_{n+1} = b_n$ for all n.

Every sequence in Γ gives us a cover of $\mathbb{H}^*$ by standard subcontinua: the family $\{[a_u, b_u] : u \in \omega^*\}$.

If two of these standard subcontinua intersect then it is (only) in the following situation: $b_u = a_v$ and $v = u + 1$. These cases will not really be important in what follows.

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Notation

If $A \in \Gamma$, say $A = \bigl\langle [a_n, b_n] : n \in \omega \bigr\rangle$, and $u \in \omega^*$ then A_u is the standard subcontinuum from the cover that contains u.

For most of the A it is actually the case that u is in a layer $L(A, u)$ of A_u; this happens if the map $\bigl\{\langle m, n\rangle : m \in [a_n, b_n]\bigr\}$ is one-to-one on no member of u.

By our technical result the $L(A, u)$ form a chain $\mathcal{C}_u$; and this is what we will use.

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Main technical result, from CH

Theorem

For every linearly ordered set T of size at most $\aleph_1$ there are a P-point u and a map $t \mapsto A_t$ from T to Γ such that $t \mapsto L(A_t, u)$ is an embedding of T into $\mathcal{C}_u$.
In addition: if T has no $\langle\omega, \omega^\star\rangle$-gaps then we can make sure that $I(T, u) = \{L(A_t, u) : t \in T\}$ is an interval in $\mathcal{C}_u$.

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Mean linear orders

Let S and T be such that

- $|S|, |T| \leq \aleph_1$
- neither S nor T has an $\langle \omega, \omega^\star \rangle$-gap
- both S and T have cofinality ω

These we call mean linear orders.

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Mean linear orders

Adjoin S as a maximum to S (and ditto for T) and apply our main technical result to the resulting ordered sets to get P-points u and v, and the corresponding embeddings.
Let us consider the layers $L(A_S, u)$ and $L(A_T, v)$.

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Mean linear orders

Because of the interval property the indecomposable continuum $L(A_S, u)$ is the closure of the F_σ-set $\bigcup_{s\in S} L(A_s, u)$ (and likewise for T and v).

Let $f : L(A_S, u) \to L(A_T, v)$ be a homeomorphism. Because the $L(A_t, u)$ are P-sets we must have $L(A_t, u) \cap f\left[\bigcup_{s\in S} L(A_s, u)\right] \neq \emptyset$ for all t (and vice versa for the $f\left[L(A_s, u)\right]$ and $\bigcup_{t\in T} L(A_t, v)$.

Use the early technical result to conclude that $f\left[\bigcup_{s\in S} L(A_s, u)\right] = \bigcup_{t\in T} L(A_t, v)$.

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It gets better

We even get, thanks to the interval property again, that the relation

$$\bigl\{\langle s,t\rangle : f\bigl[L(A_s,u)\bigr] = L(A_t,v)\bigr\}$$

is an isomorphism between final segments of S and T.

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Many mean linear orders

For a set, X, of limit ordinals in ω_1 insert a decreasing ω-sequence between α and $\alpha+1$ for all $\alpha \in X$, to form L_X, say.

Elementary: L_X and L_Y are isomorphic iff $X = Y$.

$T_X = \omega \times L_X$, ordered lexicographically.

Elementary: T_X and T_Y have isomorphic final segments iff $X = Y$.

By a happy coincidence $\aleph_1 = \mathfrak{c}$, so we have $2^{\mathfrak{c}}$ mean linear orders without isomorphic final segments.

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Oh yes, and those decomposable continua?

In each case take, in the standard continuum A_T, the closed 'interval' $J(A_T, u)$ from one end point to the layer $L(A_T, u)$.

A homeomorphism between $J(A_T, u)$ and $J(A_S, v)$ must map $L(A_T, u)$ to $L(A_S, v)$, so there.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

Alan Dow,
Some set-theory, Stone-Čech, and F-spaces, Topology and Applications, **158** (2011), 1749–1755.

Alan Dow and Klaas Pieter Hart,
On subcontinua and continuous images of $\beta\mathbb{R} \setminus \mathbb{R}$, `http://arxiv.org/abs/1401.3132`.

Klaas Pieter Hart,
The Čech-Stone compactification of the Real Line, In *Recent progress in general topology* (1992), 317–352.